Core Connections Algebra 2
Checkpoint Materials

Note to Students (and their Teachers)

Students master different skills at different speeds. No two students learn exactly the same way at the same time. At some point you will be expected to perform certain skills accurately. Most of the Checkpoint problems incorporate skills that you should have developed in previous courses. If you have not mastered these skills yet it does not mean that you will not be successful in this class. However, you may need to do some work outside of class to get caught up on them.

Starting in Chapter 2 and finishing in Chapter 12, there are 18 problems designed as Checkpoint problems. Each one is marked with an icon like the one above. After you do each of the Checkpoint problems, check your answers by referring to this section. If your answers are incorrect, you may need some extra practice to develop that skill. The practice sets are keyed to each of the Checkpoint problems in the textbook. Each has the topic clearly labeled, followed by the answers to the corresponding Checkpoint problem and then some completed examples. Next, the complete solution to the Checkpoint problem from the text is given, and there are more problems for you to practice with answers included.

Remember, looking is not the same as doing! You will never become good at any sport by just watching it, and in the same way, reading through the worked examples and understanding the steps is not the same as being able to do the problems yourself. How many of the extra practice problems do you need to try? That is really up to you. Remember that your goal is to be able to do similar problems on your own confidently and accurately. This is your responsibility. You should not expect your teacher to spend time in class going over the solutions to the Checkpoint problem sets. If you are not confident after reading the examples and trying the problems, you should get help outside of class time or talk to your teacher about working with a tutor.

Checkpoint Topics

2A. Finding the Distance Between Two Points and the Equation of a Line
2B. Solving Linear Systems in Two Variables
3A. Rewriting Expressions with Integral and Rational Exponents
3B. Using Function Notation and Identifying Domain and Range
4A. Writing Equations for Arithmetic and Geometric Sequences
4B. Solving For One Variable in an Equation with Two or More Variables
5A. Multiplying Polynomials
5B. Factoring Quadratics
6A. Multiplying and Dividing Rational Expressions
6B. Adding and Subtracting Rational Expressions
7A. Finding x- and y-Intercepts of a Quadratic Function
7B. Completing the Square to Find the Vertex of a Parabola
8A. Solving and Graphing Inequalities
8B. Solving Complicated Equations
9A. Writing and Solving Exponential Equations
9B. Finding the Equation for the Inverse of a Function
10. Rewriting Expressions with and Solving Equations with Logarithms
11. Solving Rational Equations

Checkpoint 2A

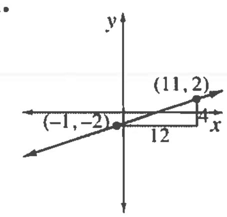
Problem 2-53

Finding the Distance Between Two Points and the Equation of a Line

Answers to problem 2-53: a: $\sqrt{45} = 3\sqrt{5} \approx 6.71$; $y = \frac{1}{2}x + 5$, b: 5; $x = 3$, c: $\sqrt{725} \approx 26.93$; $y = -\frac{5}{2}x + \frac{5}{2}$, d: 4; $y = -2$

The distance between two points is found by using the Pythagorean Theorem. The most commonly used equation of a line is $y = mx + b$ where m represents the slope of the line and b represents the y-intercept of the line. One strategy for both types of problems is to create a generic right triangle determined by the given points. The lengths of the legs of the triangle are used in the Pythagorean Theorem to find the distance. They are also used in the slope ratio to write an equation of the line. This strategy is not necessary for vertical or horizontal pairs of points, however.

Example: For the points $(-1,-2)$ and $(11,2)$, find the distance between them and determine an equation of the line through them.



Solution: Using a generic right triangle, the legs of the triangle are 12 and 4. The distance between the points is the length of the hypotenuse.

$$d^2 = 12^2 + 4^2 = 160 \Rightarrow d = \sqrt{160} = 4\sqrt{10} \approx 12.65$$

The slope of the line, $m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{4}{12} = \frac{1}{3}$. Substituting this into the equation of a line, $y = mx + b$, gives $y = \frac{1}{3}x + b$. Next substitute any point that is on the line for x and y and solve for b. Using $(11,2)$, $2 = \frac{1}{3} \cdot 11 + b$, $2 = \frac{11}{3} + b$, $b = -\frac{5}{3}$.

The equation is $y = \frac{1}{3}x - \frac{5}{3}$.

Some people prefer to use formulas that represent the generic right triangle.

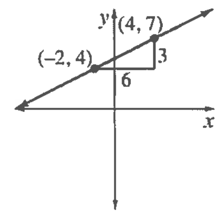
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{11-(-1)} = \frac{4}{12} = \frac{1}{3}$$

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(11-(-1))^2 + (2-(-2))^2} = \sqrt{12^2 + 4^2} = \sqrt{160}$$

Notice that $x_2 - x_1$ and $y_2 - y_1$ represent the lengths of the horizontal and vertical legs respectively.

Now we can go back and solve the original problems.

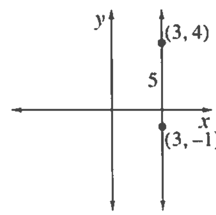
a. $d^2 = 6^2 + 3^2 \Rightarrow d^2 = 45 \Rightarrow d = \sqrt{45} = 3\sqrt{5} \approx 6.71$



$m = \frac{3}{6} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + b$

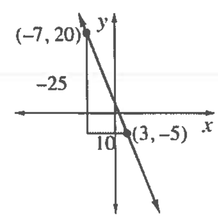
Using the point $(4,7) \Rightarrow 7 = \frac{1}{2} \cdot 4 + b \Rightarrow b = 5$.
The equation is $y = \frac{1}{2}x + 5$.

b. Since this is a vertical line, the distance is simply the difference of the y values. $d = 4-(-1) = 5$.



Vertical lines have an *undefined* slope and the equation of the line is of the form $x = k \Rightarrow x = 3$.

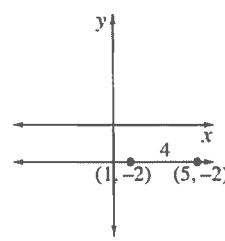
c. $d^2 = (-25)^2 + 10^2 \Rightarrow d^2 = 725 \Rightarrow d = \sqrt{725} \approx 26.93$



$m = \frac{-25}{10} = -\frac{5}{2} \Rightarrow y = -\frac{5}{2}x + b$

Using the point $(3,-5) \Rightarrow -5 = -\frac{5}{2} \cdot 3 + b \Rightarrow b = -5 + \frac{15}{2} = \frac{5}{2}$
The equation is $y = -\frac{5}{2}x + \frac{5}{2}$.

d. Since this is a horizontal line, the distance is simply the difference of the x-values. $d = 5-1 = 4$.



Horizontal lines have a slope of 0 and the equation of the line is of the form $y = k \Rightarrow y = -2$.

Here are some more to try. For each pair of points, compute the distance between them and then find an equation of the line through them.

1. $(2,3)$ and $(1,2)$
2. $(-3,-5)$ and $(-1,0)$
3. $(4,2)$ and $(8,-1)$
4. $(1,3)$ and $(5,7)$
5. $(0,4)$ and $(-1,-5)$
6. $(-3,2)$ and $(2,-3)$
7. $(4,2)$ and $(-1,-2)$
8. $(3,1)$ and $(-2,-4)$
9. $(4,1)$ and $(4,10)$
10. $(10,2)$ and $(2,22)$
11. $(-10,3)$ and $(-2,-5)$
12. $(-3,5)$ and $(12,5)$
13. $(-4,10)$ and $(-6,15)$
14. $(-6,-3)$ and $(2,10)$

Answers:

1. $\sqrt{2} \approx 1.41$; $y = x + 1$
2. $\sqrt{29} \approx 5.39$; $y = \frac{5}{2}x + \frac{5}{2}$
3. 5; $y = -\frac{3}{4}x + 5$
4. $\sqrt{32} = 4\sqrt{2} \approx 5.66$; $y = x + 2$
5. $\sqrt{82} \approx 9.06$; $y = 9x + 4$
6. $\sqrt{50} = 5\sqrt{2} \approx 7.07$; $y = -x - 1$
7. $\sqrt{41} \approx 6.40$; $y = \frac{4}{5}x - \frac{6}{5}$
8. $\sqrt{50} = 5\sqrt{2} \approx 7.07$; $y = x - 2$
9. 9; $x = 4$
10. $\sqrt{464} \approx 21.54$; $y = -\frac{5}{2}x + 27$
11. $\sqrt{128} = 8\sqrt{2} \approx 11.31$; $y = -x - 7$
12. 15; $y = 5$
13. $\sqrt{29} \approx 5.39$; $y = -\frac{5}{2}x$
14. $\sqrt{233} \approx 15.26$; $y = \frac{13}{8}x + \frac{27}{4}$

Checkpoint 2B

Problem 2-152

Solving Linear Systems in Two Variables

Answers to problem 2-152: (3, 2)

You can solve systems of equations using a variety of methods. For linear systems, you can graph the equations, use the Substitution Method, or use the Elimination Method. Each method works best with certain forms of equations. Following are some examples. Although the method that is easiest for one person may not be easiest for another, the most common methods are shown below.

Example 1: Solve the system of equations $x = 4y - 7$ and $3x - 2y = 1$.

Solution: For this, we will use the Substitution Method. Since the first equation tells us that x is equal to $4y - 7$, we can substitute $4y - 7$ for x in the second equation. This allows us to solve for y, as shown at right.

$$\begin{aligned} 3(4y-7)-2y &= 1 \\ 12y-21-2y &= 1 \\ 10y-21 &= 1 \\ 10y &= 22 \\ y &= \tfrac{22}{10} = 2.2 \end{aligned}$$

Then substitute $y = 2.2$ into either original equation and solve for x: Choosing the first equation, we get $x = 4(2.2) - 7 = 8.8 - 7 = 1.8$. To verify the solution completely check this answer in the second equation by substituting. $3(1.8) - 2(2.2) = 5.4 - 4.4 = 1$

Answer: The solution to the system is $x = 1.8$ and $y = 2.2$ or $(1.8, 2.2)$.

Example 2: Solve the system of equations $y = \frac{3}{4}x - 1$ and $y = -\frac{1}{3}x - 1$.

Solution: Generally graphing the equations is not the most efficient way to solve a system of linear equations. In this case, however, both equations are written in $y =$ form so we can see that they have the same y-intercept. Since lines can cross only at one point, no points or infinite points, and these lines have different slopes (they are not parallel or coincident), the y-intercept must be the only point of intersection and thus the solution to the system. We did not actually graph here, but we used the principles of graphs to solve the system. Substitution would work nicely as well.

Answer: $(0, -1)$

Example 3: Solve the system $x + 2y = 16$ and $x - y = 2$.

Solution: For this, we will use the Elimination Method. We can subtract the second equation from the first and then solve for y, as shown at right.

$$\begin{aligned} x + 2y &= 16 \\ -(x - y &= 2) \\ \hline 0 + 3y &= 14 \\ 3y &= 14 \\ y &= \tfrac{14}{3} \end{aligned}$$

We then substitute $y = \frac{14}{3}$ into either original equation and solve for x. Choosing the second equation, we get $x - \frac{14}{3} = 2$, so $x = 2 + \frac{14}{3} = \frac{20}{3}$. Checking our solution can be done by substituting both values into the first equation.

Answer: The solution to the system is $(\frac{20}{3}, \frac{14}{3})$.

Example 4: Solve the system $x + 3y = 4$ and $3x - y = 2$.

Solution: For this, we will use the Elimination Method, only we will need to do some multiplication first. If we multiply the second equation by 3 and add the result to the first equation, we can eliminate y and solve for x, as shown at right.

$$\begin{aligned} x + 3y &= 4 \\ +\ 9x - 3y &= 6 \\ \hline 10x \quad &= 10 \\ x &= 1 \end{aligned}$$

We can then find y by substituting $x = 1$ into either of the original equations. Choosing the second, we get $3(1) - y = 2$, which solves to yield $y = 1$. Again, checking the solution can be done by substituting both values into the first equation.

Answer: The solution to this system is $(1, 1)$.

Now we can return to the original problem.

Solve the following system of linear equations in two variables.

$$5x - 4y = 7$$
$$2y + 6x = 22$$

For this system, you can use either the Substitution or the Elimination Method, but each choice will require a little bit of work to get started.

Substitution Method:

Before we can substitute, we need to isolate one of the variables. In other words, we need to solve one of the equations for either x or for y. If we solve the second equation for y, it becomes $y = 11 - 3x$. Now we substitute $11 - 3x$ for y in the first equation and solve for x, as shown at right.

$$\begin{aligned} 5x - 4(11 - 3x) &= 7 \\ 5x - 44 + 12x &= 7 \\ 17x - 44 &= 7 \\ 17x &= 51 \\ x &= 3 \end{aligned}$$

Then we can substitute the value for x into one of the original equations to find y. Thus we find that $2y + 6(3) = 22 \Rightarrow 2y = 22 - 18 = 4 \Rightarrow y = \frac{4}{2} = 2$.

Elimination Method:

Before we can eliminate a variable, we need to rearrange the second equation so that the variables line up, as shown at right. Now we see that we can multiply the second equation by 2 and add the two equations to eliminate y and solve for x, as shown below right.

$$\begin{aligned} 5x - 4y &= 7 \\ 6x + 2y &= 22 \end{aligned}$$

$$\begin{array}{r} 5x - 4y = 7 \\ +\ 12x + 4y = 44 \\ \hline 17x \qquad = 51 \\ x = 3 \end{array}$$

We can then substitute $x = 3$ into the first equation to get $5(3) - 4y = 7$. Simplifying and solving, we get $-4y = -8$ and thus $y = 2$.

Answer: $(3, 2)$

Here are some more to try. Find the solution to these systems of linear equations. Use the method of your choice.

1. $y = 3x - 1$
   $2x - 3y = 10$

2. $x = -0.5y + 4$
   $8x + 3y = 31$

3. $2y = 4x + 10$
   $6x + 2y = 10$

4. $3x - 5y = -14$
   $x + 5y = 22$

5. $4x + 5y = 11$
   $2x + 6y = 16$

6. $x + 2y = 5$
   $x + y = 5$

7. $2x - 3 = y$
   $x - y = -4$

8. $y + 2 = x$
   $3x - 3y = x + 14$

9. $2x + y = 7$
   $x + 5y = 12$

10. $y = \frac{3}{5}x - 2$
    $y = \frac{x}{10} + 1$

11. $2x + y = -2x + 5$
    $3x + 2y = 2x + 3y$

12. $4x - 3y = -10$
    $x = \frac{1}{4}y - 1$

13. $4y = 2x$
    $2x + y = \frac{x}{2} + 1$

14. $3x - 2y = 8$
    $4y = 6x - 5$

15. $4y = 2x - 4$
    $3x + 5y = -3$

16. $\frac{x}{3} + \frac{4y}{3} = 300$
    $3x - 4y = 300$

Answers:

1. $(-1, -4)$
2. $(\frac{7}{2}, 1)$
3. $(0, 5)$
4. $(2, 4)$
5. $(-1, 3)$
6. $(5, 0)$
7. $(7, 11)$
8. $(-8, -10)$
9. $(\frac{23}{9}, \frac{17}{9})$
10. $(6, 1.6)$
11. $(1, 1)$
12. $(-\frac{1}{4}, 3)$
13. $(\frac{1}{2}, \frac{1}{4})$
14. no solution
15. $(\frac{4}{11}, -\frac{9}{11})$
16. $(300, 150)$

Checkpoint 3A

Problem 3-67

Expressions with Integral and Rational Exponents

Answers to problem 3-67:

a: $x^{1/5}$, b: x^{-3}, c: $\sqrt[3]{x^2}$, d: $x^{-1/2}$, e: $\frac{1}{xy^8}$, f: $\frac{1}{m^3}$, g: $xy^3\sqrt{x}$, h: $\frac{1}{81x^6y^{12}}$

The following properties are useful for rewriting expressions with integral (positive or negative whole numbers) or rational (fractional) exponents.

$x^0 = 1$ Examples: $2^0 = 1$, $(-3)^0 = 1$, $(\frac{1}{4})^0 = 1$ (Note that 0^0 is undefined.)

$x^{-n} = \frac{1}{x^n}$ Examples: $x^{-12} = \frac{1}{x^{12}}$, $y^{-4} = \frac{1}{y^4}$, $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$\frac{1}{x^{-n}} = x^n$ Examples: $\frac{1}{x^{-5}} = x^5$, $\frac{1}{x^{-2}} = x^2$, $\frac{1}{3^{-2}} = 3^2 = 9$

$x^{a/b} = (x^a)^{1/b} = (\sqrt[b]{x})^a$ Examples: $5^{1/2} = \sqrt{5}$

or : $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$

$x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$ $4^{2/3} = \sqrt[3]{4^2} = \sqrt[3]{16} = 2\sqrt[3]{2}$

$x^a x^b = x^{(a+b)}$ Examples: $x^7x^2 = x^9$, $y^{-4}y = y^{-3}$, $2^3 2^2 = 2^5 = 32$

$(x^a)^b = x^{ab}$ Examples: $(x^2)^3 = x^6$, $(a^6b^4)^{1/2} = a^3b^2$, $(3^3)^3 = 3^9 = 19683$

Now we can go back and solve the original problems.

a. Using the fourth property above, $\sqrt[5]{x} = x^{1/5}$.

b. Using the second property above, $\frac{1}{x^3} = x^{-3}$.

c. Using the fourth property above, $x^{2/3} = \sqrt[3]{x^2}$.

d. Using the second and fourth properties above, $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$.

e. Using the second property above, $x^{-1}y^{-8} = \frac{1}{xy^8}$.

f. Using the second and sixth properties above, $(m^2)^{-3/2} = m^{-3} = \frac{1}{m^3}$.

g. Using the fourth, fifth, and sixth properties above, $(x^3y^6)^{1/2} = x^{3/2}y^3 = x^1x^{1/2}y^3 = xy^3\sqrt{x}$.

h. Using the second and sixth properties above, $(9x^3y^6)^{-2} = 9^{-2}x^{-6}y^{-12} = \frac{1}{81x^6y^{12}}$.

Here are some exercises to try. For problems 1 through 12, rewrite each expression. For problems 13 through 24, simplify each expression. You should not need a calculator for any of these problems.

1. x^{-5}	2. m^0	3. 4^{-1}
4. $\sqrt[3]{y}$	5. $\frac{1}{c^4}$	6. $\frac{1}{b^{-2}}$
7. $12^{1/12}$	8. $z^{-3/4}$	9. $\frac{1}{(\sqrt[6]{7})^5}$
10. 0^0	11. $9^{1/2}$	12. $\sqrt[5]{a^3}$
13. $(f^3)\sqrt[3]{f^3}$	14. $(\frac{1}{27})^{-1/3}$	15. $(v^2g^{3/4})^8$
16. $(\frac{1}{q^6})^7$	17. $d^{-9}d^{-4}$	18. $(3xw^4)^{-2}$
19. $(u^3r^{-4})^{-2}$	20. $n^3(n^2)^5$	21. $4(\sqrt{4})^4$
22. $6(k^{1/2}t^5)^2$	23. $p^{15}p^{-15}$	24. $h^8s^{12}(\sqrt[8]{h})(s^{1/4})$

Answers:

1. $\frac{1}{x^5}$	2. 1	3. $\frac{1}{4}$
4. $y^{1/3}$	5. c^{-4}	6. b^2
7. $\sqrt[12]{12}$	8. $\frac{1}{z^{3/4}}$	9. $7^{-1/6}$
10. undefined	11. 3	12. $a^{3/5}$
13. f^4	14. 3	15. $v^{16}g^6$
16. q^{-42}	17. $d^{-13} = \frac{1}{d^{13}}$	18. $\frac{1}{9x^2w^{12}}$
19. $\frac{r^8}{u^6}$	20. n^{13}	21. 64
22. $6kt^{10}$	23. 1	24. hs^3

Checkpoint 3B

Problem 3-116

Using Function Notation and Identifying Domain and Range

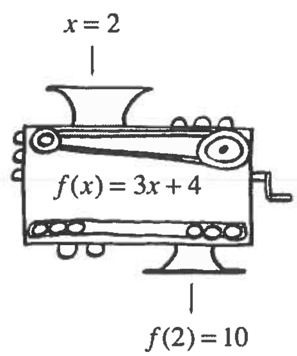
Answers to problem 3-116: Domain: all x; Range: $y \geq 0$

a. $g(-5) = 8$ b. $g(a+1) = 2a^2 + 16a + 32$

c. $x = 1$ or $x = -7$ d. $x = -3$

An equation is called a function if there exists *no more than one* output for each input. If an equation has two or more outputs for a single input value, it is not a function. The set of possible inputs of a function is called the domain, while the set of all possible outputs of a function is called the range.

Functions are often given names, most commonly "*f*," "*g*," or "*h*." The notation $f(x)$ represents the output of a function, named *f*, when *x* is the input. It is pronounced "*f* of *x*." The notation *g*(2), pronounced "*g* of 2," represents the output of function *g* when $x = 2$.

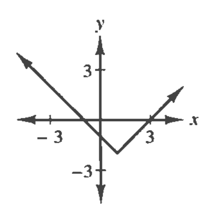


Similarly, the function $y = 3x + 4$ and $f(x) = 3x + 4$ represent the *same function*. Notice that the notation is interchangeable, that is $y = f(x)$. In some textbooks, $3x + 4$ is called the **rule** of the function. The graph of $f(x) = 3x + 4$ is a line extending forever in both the *x* (horizontal) and the *y* (vertical) directions, so the domain and range of $f(x) = 3x + 4$ are all real numbers.

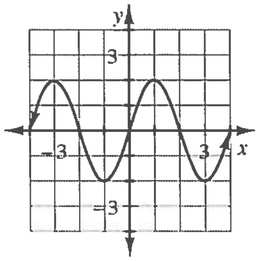
Examples 1 through 3: For each function below, give the domain and range. Then calculate $f(2)$ and solve $f(x) = 3$.

Example 1: $f(x) = |x - 1| - 2$

Solution: We start by graphing the function, as shown at right. Since we can use any real number for *x* in this equation, the domain is all real numbers. The smallest possible result for y is –2, so the range is $y \geq -2$. By looking at the graph or substituting $x = 2$ into the equation, $f(2) = |2 - 1| - 2 = -1$. To solve $f(x) = 3$, find the points where the horizontal line $y = 3$ intersects the graph or solve the equation $3 = |x - 1| - 2$, which yields $x = -4$ or $x = 6$.



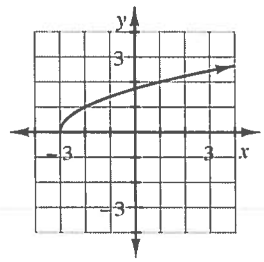
Example 2: $f(x)$ is given by the graph below.



The arrows indicate that the graph continues indefinitely right and left and we see no disruption in the smooth function, so the domain is all real numbers. All of the y-values fall between −2 and 2, so the range is $-2 \le y \le 2$. We can see from the graph that when $x = 2$, the value of the function appears to be 0, or $f(2) \approx 0$. Since $-2 \le y \le 2$, the value of the function never gets as high as 3, so $f(x) = 3$ has no solution.

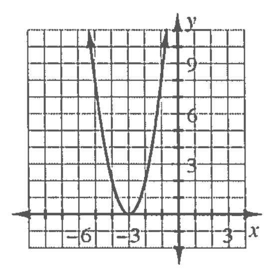
Example 3: $f(x) = \sqrt{x+3}$

Solution: Again, we start by making a graph of the function, which is shown at right. Since the square root of a negative number does not exist, we can only use x-values of −3 or larger. Thus, the domain is $x \ge -3$. We can see from the graph and the equation that the smallest possible y-value is zero, so the range is $y \ge 0$. Looking at the graph gives an approximate answer when $x = 2$ of $y \approx 2.25$. Or, by substituting $x = 2$ into the equation, we get $f(2) = \sqrt{2+3} = \sqrt{5}$. To solve $f(x) = 3$, find the point where $y = 3$ intersects the graph or solve $3 = \sqrt{x+3}$, which gives $x = 6$.



Now we can go back and solve the original problem.

The graph is a parabola opening upward with vertex $(-3,0)$, as shown at right. Thus, the domain is all real numbers and the range is $y \ge 0$.



$g(-5) = -2(-5+3) = 2(-2)^2 = 8$

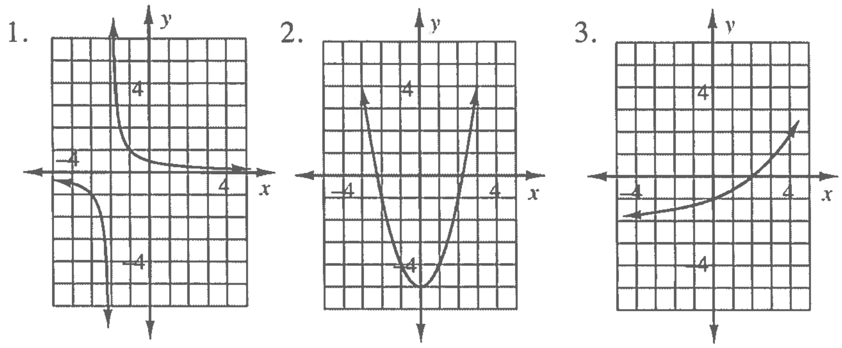
$g(a+1) = 2(a+1+3)^2 = 2(a+4)^2$
$= 2(a^2 + 8a + 16) = 2a^2 + 16a + 32$

If $g(x) = 32$, then $32 = 2(x+3)^2$. Dividing both sides by 2, we get $16 = (x+3)^2$. Taking the square root of both sides gives $\pm 4 = x + 3$, which leads to the values $x = 1$ or -7.

If $g(x) = 0$, then $0 = 2(x+3)^2$. Diving both sides by two or applying the Zero Product Property gives $0 = (x+3)^2$ and then $0 = x + 3$. Thus $x = -3$.

Here are some more to try.

For each graph in problems 1 through 3, describe the domain and range.



4. If $f(x) = 3 - x^2$, calculate $f(5)$ and $f(3a)$.

5. If $g(x) = 5 - 3x^2$, calculate $g(-2)$ and $g(a+2)$.

6. If $f(x) = \frac{x+3}{2x-5}$, calculate $f(2)$ and $f(2.5)$.

7. If $f(x) = x^2 + 5x + 6$, solve $f(x) = 0$.

8. If $g(x) = 3(x-5)^2$, solve $g(x) = 27$.

9. If $f(x) = (x+2)^2$, solve $f(x) = 27$.

Answers:

1. Domain: $x \neq -2$, Range: $y \neq 0$

2. Domain: all real numbers, Range: $y \geq -5$

3. Domain: all real numbers, Range: $y > -2$

4. $f(5) = -22$, $f(3a) = 3 - 9a^2$

5. $g(-2) = -7$, $g(a+2) = -3a^2 - 12a - 7$

6. $f(2) = -5$, not possible

7. $x = -2$ or $x = -3$

8. $x = 8$ or $x = 2$

9. $x = -2 \pm \sqrt{27}$

Checkpoint 4A

Problem 4-42

Writing Equations for Arithmetic and Geometric Sequences

Answers to problem 4-42: a: E $t(n) = -2 + 3n$; R $t(0) = -2,\ t(n+1) = t(n) + 3$,
b: E $t(n) = 6(\frac{1}{2})^n$; R $t(0) = 6,\ t(n+1) = \frac{1}{2}t(n)$, c: $t(n) = 10 - 7n$, d: $t(n) = 5(1.2)^n$,
e: $t(4) = 1620$

An ordered list of numbers such as: $4, 9, 16, 25, 36, \ldots$ creates a sequence. The numbers in the sequence are called terms. One way to identify and label terms is to use function notation. For example, if $t(n)$ is the name of the sequence above, the initial value is 4 and the second term after the initial value is 16. This is written $t(0) = 4$ and $t(2) = 16$. Some books use subscripts instead of function notation. In this case $t_0 = 4$ and $t_2 = 16$. The initial value is *not* part of the sequence. It is only a reference point and is useful in writing a rule for the sequence. When writing a sequence, start by writing the first term after the initial value, $t(1)$ or t_1 .

Arithmetic sequences have a common difference between the terms. The rule for the values in an arithmetic sequences can be found by $t(n) = a + dn$ where a = the initial value, d = the common difference and n = the number of terms after the initial value.

Geometric sequences have a common ratio between the terms. The rule for the values in a geometric sequence may be found by $t(n) = ar^n$ where a = the initial value, r = the common ratio and n = the number of terms after the initial value.

Example 1: Find a rule for the sequence: $-2, 4, 10, 16, \ldots$

Solution: There is a common difference between the terms $(d = 6)$ so it is an arithmetic sequence. Work backward to find the initial value: $a = -2 - 6 = -8$.
Now use the general rule: $t(n) = a + dn = -8 + 6n$.

Example 2: Find a rule for the sequence: $81, 27, 9, 3, \ldots$

Solution: There is a common ratio between the terms $(r = \frac{1}{3})$ so it is a geometric sequence.
Work backward to find the initial value: $a = 81 \div \frac{1}{3} = 243$.
Now use the general rule: $t(n) = ar^n = 243(\frac{1}{3})^n$.

A rule such as $t(n) = 5 - 7n$ is called an explicit rule because any term can be found by substituting the appropriate value for n into the rule. To find the 10^{th} term after the initial value, $t(10)$, substitute 10 for n. $t(10) = 5 - 7(10) = -65$.

A second way to find the terms in a sequence is by using a recursive formula. A recursive formula tells first term or the initial value and how to get from one term to the next.

Example 3: (using subscript notation)
Write the first five terms of the sequence determined by: $b_1 = 8,\ b_{n+1} = b_n \cdot \frac{1}{2}$

Solution: $b_1 = 8$ tells you the first term and $b_{n+1} = b_n \cdot \frac{1}{2}$ tells you to multiply by $\frac{1}{2}$ to get from one term to the next.

$b_1 = 8$ $\quad b_2 = b_1 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$ $\quad b_3 = b_2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$

$b_4 = b_3 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$ $\quad b_5 = b_4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

The sequence is: $8, 4, 2, 1, \frac{1}{2}, \ldots$

Now we can go back and solve the original problems.

a. It is an arithmetic sequence $(d = 3)$. Working backward the initial value is $1 - 3 = -2$. Using the general formula the explicit rule: $t(n) = a + dn = -2 + 3d$. A possible recursive rule is $t(1) = 1, t(n+1) = t(n) + 3$.

b. It is a geometric sequence $(r = \frac{1}{2})$. Working backward the initial value is $3 \div \frac{1}{2} = 6$. Using the general formula for the explicit rule: $t(n) = ar^n = 6(\frac{1}{2})^n$. A possible recursive rule is $t(0) = 6, t(n+1) = \frac{1}{2}t(n)$.

c. $t(2)$ is halfway between $t(1)$ and $t(3)$ so $t(2) = 10$. This means $d = -7$ and the initial value is 24. Using the general formula the explicit rule: $t(n) = a + dn = 24 - 7d$.

d. The common ratio $r = \frac{8.64}{7.2} = 1.2$ so $t(1) = \frac{7.2}{1.2} = 6, t(0) = \frac{6}{1.2} = 5$. Using an initial value of 5 and a common ratio of 1.2 in the general formula for the explicit rule: $t(n) = ar^n = 5(1.2)^n$.

e. The common difference is the difference in the values divided by the number of terms. $d = \frac{t(12)-t(7)}{12-7} = \frac{116-1056}{5} = -188$. Working backward three terms: $t(4) = 1056 - 3(-188) = 1620$.

Here are some more to try.

Write the first 6 terms of each sequence.

1. $t(n) = 5n + 2$
2. $t(n) = 6(-\frac{1}{2})^n$
3. $t(n) = -15 + \frac{1}{2}n$
4. $t_n = -3 \cdot 3^{n-1}$
5. $t(1) = 3,\ t(n+1) = t(n) - 5$
6. $t_1 = \frac{1}{3},\ t_{n+1} = \frac{1}{3}t_n$

For each sequence, write an explicit and recursive rule.

7. 10, 50, 250, 1250, ...
8. 4, 8, 12, 16, ...
9. −2, 5, 12, 19, ...
10. 16, 4, 1, $\frac{1}{4}$, ...
11. −12, 6, −3, $\frac{3}{2}$, ...
12. $\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$, ...

For each sequence, write an explicit rule.

13. A geometric sequence

n	$t(n)$
0	
1	15
2	45
3	
4	

14. An arithmetic sequence

n	$t(n)$
0	27
1	15
2	
3	
4	

15. An arithmetic sequence

n	$t(n)$
1	
2	$3\frac{1}{3}$
3	
4	
5	$4\frac{1}{3}$

16. A geometric sequence

n	$t(n)$
1	
2	
3	−24
4	48
5	

Solve each problem.

17. An arithmetic sequence has $t(3)=52$ and $t(10)=108$. Find a rule for $t(n)$ and find $t(100)$.

18. An arithmetic sequence has $t(1)=-17$, $t(2)=-14$ and $t(n)=145$. What is the value of n?

19. An arithmetic sequence has $t(61)=810$ and $t(94)=1239$. Find a rule for $t(n)$.

20. A geometric sequence has $t(4)=12$ and $t(7)=324$. Find the common ratio and a rule for $t(n)$.

Answers:

1. $7, 12, 17, 22, 27, 32$
2. $-3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}$
3. $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}, -12$
4. $-3, -9, -27, -81, -243, -729$
5. $3, -2, -7, -12, -17, -22$
6. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}$

Rules for problems 7 through 20 may vary.

7. $t(n)=2\cdot 5^n$; $t(0)=2,\ t(n+1)=5t(n)$
8. $t(n)=4n$; $t(0)=0,\ t(n+1)=t(n)+4$
9. $t(n)=-9+7n$; $t(0)=-9,\ t(n+1)=t(n)+7$
10. $t(n)=64(\frac{1}{4})^n$; $t(0)=64,\ t(n+1)=\frac{1}{4}t(n)$
11. $t(n)=24(-\frac{1}{2})^n$; $t(0)=24,\ t(n+1)=-\frac{1}{2}t(n)$
12. $t(n)=1-\frac{1}{6}n$; $t(0)=1,\ t(n+1)=t(n)-\frac{1}{6}$
13. $t(n)=5\cdot 3^n$
14. $t(n)=27-12n$
15. $t(n)=2\frac{2}{3}+\frac{1}{3}n$
16. $t(n)=3(-2)^n$
17. $t(n)=28+8n$; $t(100)=828$
18. $n=55$
19. $t(n)=17+13n$
20. $t(n)=\frac{4}{27}(3)^n$

Checkpoint 4B

Problem 4-87

Solving for One Variable in an Equation with Two or More Variables

Answers to problem 4-87: a: $y=\frac{1}{3}x-4$, b: $y=\frac{6}{5}x-\frac{1}{5}$, c: $y=(x+1)^2+4$, d: $y=x^2+4x$

When we want to solve for one variable in an equation with two or more variables it usually helps to start by simplifying, such as removing parentheses and fractions. Next isolate the desired variable in the same way as you solve an equation with only one variable. Here are two examples.

Example 1: Solve $\frac{x-3y}{4}+2(x+1)=7$ for y.

Solution: First multiply all terms by 4 to remove the fraction and then simplify, as shown at right. Then, to isolate y, we subtract $9x$ from both sides to get $-3y=-9x+20$. Dividing both sides by -3 results in $y=3x-\frac{20}{3}$.

$$(4)\frac{x-3y}{4}+4(2)(x+1)=4(7)$$
$$x-3y+8x+8=28$$
$$9x-3y=20$$

Answer: $y=3x-\frac{20}{3}$

Example 2: Solve $x+2\sqrt{y+1}=3x+4$ for y.

Solution: First, we isolate the radical by subtracting x from both sides to get $2\sqrt{1+y}=2x+4$ and then dividing both sides by 2 to get $\sqrt{1+y}=x+2$. Then, we remove the radical by squaring both sides, as shown at right. Lastly, we isolate y by subtracting 1 from both sides of the equation.

$$(\sqrt{y+1})^2=(x+2)^2$$
$$y+1=(x+2)(x+2)$$
$$y+1=x^2+4x+4$$

Answer: $y=x^2+4x+3$

Now we can go back and solve the original problems.

a.
$$x-3(y+2)=6$$
$$x-3y-6=6$$
$$x-3y=12$$
$$-3y=-x+12$$
$$y=\frac{-x+12}{-3} \text{ or } y=\frac{1}{3}x-4$$

b.
$$\frac{6x-1}{y}-3=2$$
$$\frac{6x-1}{y}=5$$
$$(y)\frac{6x-1}{y}=5(y)$$
$$6x-1=5y$$
$$y=\frac{6x-1}{5} \text{ or } y=\frac{6}{5}x-\frac{1}{5}$$

c.
$$\sqrt{y-4}=x+1$$
$$(\sqrt{y-4})^2=(x+1)^2$$
$$y-4=(x+1)^2$$
$$y=(x+1)^2+4 \text{ or } x^2+2x+5$$

d.
$$\sqrt{y+4}=x+2$$
$$(\sqrt{y+4})^2=(x+2)^2$$
$$y+4=x^2+4x+4$$
$$y=x^2+4x$$

Here are some more to try. Solve each equation for y.

1. $2x-5y=7$
2. $2(x+y)+1=x-4$
3. $4(x-y)+12=2x-4$
4. $x=\frac{1}{5}y-2$
5. $x=y^2+1$
6. $\frac{5x+2}{y}-1=5$
7. $\sqrt{y+3}=x-2$
8. $(y+2)^2=x^2+9$
9. $\frac{x+2}{4}+\frac{4-y}{2}=3$
10. $\sqrt{2y+1}=x+3$
11. $x=\frac{2}{4-y}$
12. $x=\frac{y+1}{y-1}$

Answers:

1. $y=\frac{2}{5}x-\frac{7}{5}$
2. $y=-\frac{1}{2}x-\frac{5}{2}$
3. $y=\frac{1}{2}x+4$
4. $y=5x+10$
5. $y=\pm\sqrt{x-1}$
6. $y=\frac{5}{6}x+\frac{1}{3}$
7. $y=x^2-4x+1$
8. $y=\pm\sqrt{x^2+9}-2$
9. $y=\frac{1}{2}x-1$
10. $y=\frac{1}{2}x^2+3x+4$ or $y=\frac{1}{2}(x+4)(x+2)$
11. $y=\frac{4x-2}{x}$ or $y=4-\frac{2}{x}$
12. $y=\frac{x+1}{x-1}$

Checkpoint 5A

Problem 5-49

Multiplying Polynomials

Answers to problem 5-49:

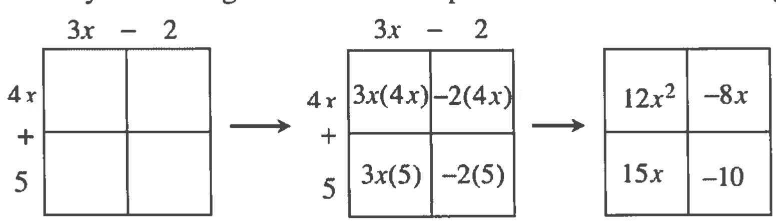
a. $2x^3 + 2x^2 - 3x - 3$ b. $x^3 - x^2 + x + 3$

c. $2x^2 + 12x + 18$ d. $4x^3 - 8x^2 - 3x + 9$

The product of polynomials can be found by using the Distributive Property. Using generic rectangles or, in the case of multiplying two binomials, the FOIL Method can help you to keep track of the terms to be sure that you are multiplying correctly.

Example: Multiply $(3x - 2)(4x + 5)$.

Solution 1: When multiplying binomials, such as $(3x - 2)(4x + 5)$, you can use a generic rectangle. You consider the terms of your original binomials as the dimensions (length and width) of the rectangle. To find the area of each piece, you multiply the terms that represent the length and width of that piece. To get your final answer, you add the areas of each of the interior pieces and simplify by combining like terms. This process is shown in the diagram below.



$12x^2 + 15x - 8x - 10 = 12x^2 + 7x - 10$

Solution 2: You might view multiplying binomials with generic rectangles as a form of double distribution. The $4x$ is distributed across the first row of the generic rectangle. Then the 5 is distributed across the second row of the generic rectangle. Some people write it this way:

$$(3x - 2)(4x + 5) = (3x - 2)4x + (3x - 2)5 = 12x^2 - 8x + 15x - 10 = 12x^2 + 7x - 10$$

Solution 3: Another approach to multiplying binomials is to the FOIL Method. This method uses the mnemonic "FOIL," which is an acronym for First, Outside, Inside, Last, to help you remember which terms to multiply.

F. Multiply the FIRST terms of each binomial. $(3x)(4x)=12x^2$

O. Multiply the OUTSIDE terms. $(3x)(5)=15x$

I. Multiply the INSIDE terms. $(-2)(4x)=-8x$

L. Multiply the LAST terms of each binomial . $(-2)(5)=-10$

Finally combine like terms to get $12x^2+15x-8x-10=12x^2+7x-10$.

Answer: $12x^2+7x-10$

Now we can go back and solve the original problems.

a: $(x+1)(2x^2-3)$

Solution: We can use the FOIL Method here. Multiplying the *first* terms, we get $(x)(2x^2)=2x^3$. Multiplying the *outside* terms, we get $(x)(-3)=-3x$. Multiplying the *inside* terms, we get $(1)(2x^2)=2x^2$. Multiplying the *last* terms, we get $(1)(-3)=-3$. Adding these results, we get $2x^3-3x+2x^2-3$. Generally, answers are expressed in with terms in order of decreasing powers of x, so we rearrange terms for the answer.

Answer: $2x^3+2x^2-3x-3$

b: $(x+1)(x-2x^2+3)$

Solution: This is a good problem for a generic rectangle, as shown at right. After calculating the area of each individual cell, we find our expression by adding them together to get $x^3-2x^2+x^2+3x-2x+3$. Then we combine like terms to get a simplified answer.

	x^2	$-2x$	$+3$
x	x^3	$-2x^2$	$3x$
$+1$	x^2	$-2x$	3

Answer: x^3-x^2+x+3

c: $2(x+3)^2$

Solution: Here we write out the factors and use the Distributive Property, as shown in the solution at right.

$$\begin{aligned} &2(x+3)(x+3) \\ &= (2x+6)(x+3) \\ &= (2x+6)(x)+(2x+6)(3) \\ &= 2x^2+6x+6x+18 \\ &= 2x^2+12x+18 \end{aligned}$$

Answer: $2x^2+12x+18$

d: $(x+1)(2x-3)^2$

Solution: Write out the factors. Multiply two of the factors together and then multiply that result by the third factor. This process is shown at right.

$$\begin{aligned} &(x+1)(2x-3)(2x-3) \\ &= (2x^2-x-3)(2x-3) \\ &= 4x^3-6x^2-2x^2+3x-6x+9 \\ &= 4x^3-8x^2-3x+9 \end{aligned}$$

Answer: $4x^3-8x^2-3x+9$

Here are some more to try. Multiply and simplify.

1. $(2x+3)(x-7)$
2. $(4x-2)(3x+5)$
3. $(x-2)(x^2+3x+5)$
4. $(x+8)(x-12)$
5. $4(3x-5)^2$
6. $(2x+y)(2x-y)$
7. $(2x+3)^2$
8. $(5x-8)(2x+7)$
9. $(x+3)(x^2-4x+7)$
10. $(x+7)(x-11)$
11. $-8x^2(5x^2+7)$
12. $(2x+y)(x+1)^2$

Answers:

1. $2x^2-11x-21$
2. $12x^2+14x-10$
3. x^3+x^2-x-10
4. $x^2-4x-96$
5. $36x^2-120x+100$
6. $4x^2-y^2$
7. $4x^2+12x+9$
8. $10x^2+19x-56$
9. $x^3-x^2-5x+21$
10. $x^2-4x-77$
11. $-40x^4-56x^2$
12. $2x^3+4x^2+2x+x^2y+2xy+y$

Checkpoint 5B

Problem 5-100

Factoring Quadratics

Answers to problem 5-100:

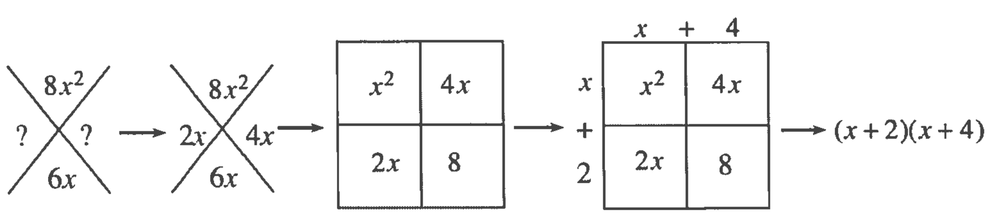
a.	$(2x+1)(2x-1)$	b.	$(2x+1)^2$
c.	$(2y+1)(y+2)$	d.	$(3m+1)(m-2)$

Factoring quadratics means changing the expression into a product of factors or to find the dimensions of the generic rectangle that represents the quadratic. You can use Diamond Problems with generic rectangles or just guess and check with FOIL Method or the Distributive Property to factor.

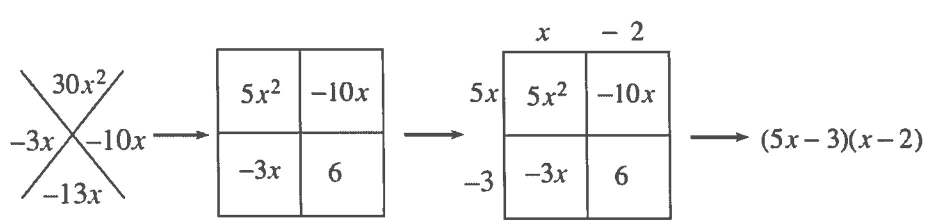
Here are some examples using Diamond Problems and generic rectangles:

Example 1: Factor x^2+6x+8.

Solution: Multiply the x^2-term by the constant term and place the result in the top of the diamond. This will be the product of the two sides of the diamond. Then place the x-term at the bottom of the diamond. This will be the sum of the sides. Then find two terms that multiply to give the top term in the diamond and add to give the bottom term in the diamond, in this case $2x$ and $6x$. This tells us how the x-term is split in the generic rectangle. Once we have the area of the generic rectangle we can find the dimensions by looking for common factors among rows and columns. Study the example below.

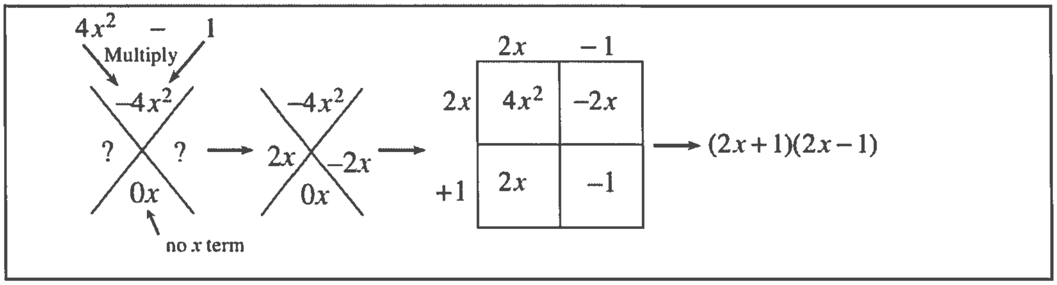


Example 2: Factor $5x^2-13x+6$.

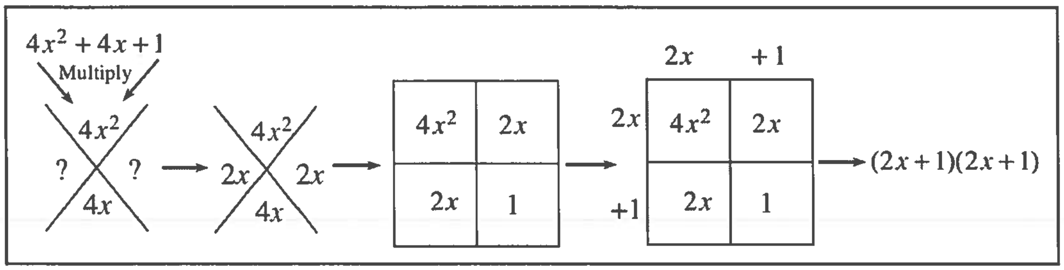


Now we can go back and solve the original problems.

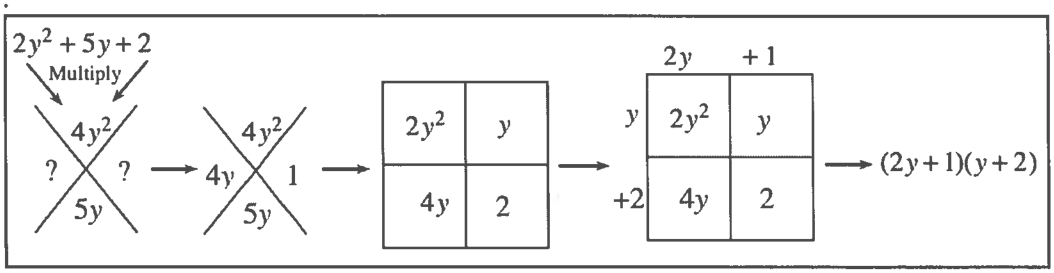
a.



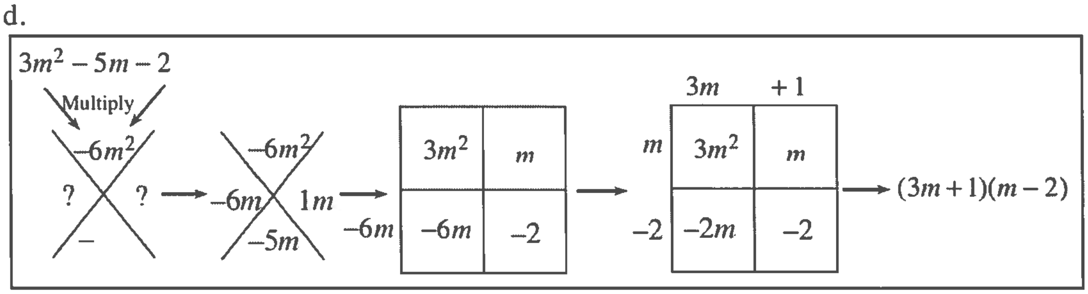
b.



c.



d.



Here are some more to try. Factor each expression.

1. $2x^2 + 7x - 4$
2. $7x^2 + 13x - 2$
3. $3x^2 + 11x + 10$
4. $x^2 + 5x - 24$
5. $2x^2 + 5x - 7$
6. $3x^2 - 13x + 4$
7. $64x^2 + 16x + 1$
8. $5x^2 + 12x - 9$
9. $8x^2 + 24x + 10$
10. $6x^3 + 31x^2 + 5x$

Answers:

1. $(x+4)(2x-1)$
2. $(7x-1)(x+2)$
3. $(3x+5)(x+2)$
4. $(x+8)(x-3)$
5. $(2x+7)(x-1)$
6. $(3x-1)(x-4)$
7. $(8x+1)^2$
8. $(5x-3)(x+3)$
9. $2(4x^2 + 12x + 5) = 2(2x+1)(2x+5)$
10. $x(6x^2 + 31x + 5) = x(6x+1)(x+5)$

Checkpoint 6A

Problem 6-73

Multiplying and Dividing Rational Expressions

Answers to problem 6-73: a: $\frac{x+3}{x+4}$, b: $\frac{1}{x(x+2)}$

Multiplication or division of rational expressions follows the same procedure used with numerical fractions. However, it is often necessary to factor the polynomials in order to simplify them. Factors that are the same in the numerator and denominator are equal to 1. For example: $\frac{x^2}{x^2}=1$, $\frac{(x+2)}{(x+2)}=1$ and $\frac{(3x-2)}{(3x-2)}=1$ but $\frac{5-x}{x-5}=\frac{-(x-5)}{x-5}=-1$.

When dividing rational expressions, change the problem to multiplication by inverting (flipping) the second expression (or any expression that follows a division sign) and completing the process as you do for multiplication.

In both cases, the simplification is only valid provided that the denominator is not equal to zero. See the examples below.

Example 1: Multiply $\frac{x^2+6x}{(x+6)^2}\cdot\frac{x^2+7x+6}{x^2-1}$ and simplify the result.

Solution:

After factoring, the expression becomes: $\frac{x(x+6)}{(x+6)(x+6)}\cdot\frac{(x+6)(x+1)}{(x+1)(x-1)}$

After multiplying, reorder the factors: $\frac{(x+6)}{(x+6)}\cdot\frac{(x+6)}{(x+6)}\cdot\frac{x}{(x-1)}\cdot\frac{(x+1)}{(x+1)}$

Since $\frac{(x+6)}{(x+6)}=1$ and $\frac{(x+1)}{(x+1)}=1$, simplify: $1\cdot 1\cdot\frac{x}{x-1}\cdot 1 \Rightarrow \frac{x}{x-1}$ for $x\neq -6, -1$, or 1.

Example 2: Divide $\frac{x^2-4x-5}{x^2-4x+4}\div\frac{x^2-2x-15}{x^2+4x-12}$ and simplify the result.

Solution:

First change to a multiplication expression by inverting (flipping) the second fraction: $\frac{x^2-4x-5}{x^2-4x+4}\cdot\frac{x^2+4x-12}{x^2-2x-15}$

After factoring, the expression is: $\frac{(x-5)(x+1)}{(x-2)(x-2)}\cdot\frac{(x+6)(x-2)}{(x-5)(x+3)}$

Reorder the factors (if you need to): $\frac{(x-5)}{(x-5)}\cdot\frac{(x-2)}{(x-2)}\cdot\frac{(x+1)}{(x-2)}\cdot\frac{(x+6)}{(x+3)}$

Since $\frac{(x-5)}{(x-5)}=1$ and $\frac{(x-2)}{(x-2)}=1$, simplify: $\frac{(x+1)}{(x-2)}\cdot\frac{(x+6)}{(x+3)}$

Thus, $\frac{x^2-4x-5}{x^2-4x+4}\div\frac{x^2-2x-15}{x^2+4x-12}=\frac{(x+1)}{(x-2)}\cdot\frac{(x+6)}{(x+3)}$ or $\frac{x^2+7x+6}{x^2+x-6}$ for $x\neq -3$, 2, or 5.

Now we can go back and solve the original problems.

a. $\frac{x^2-16}{(x-4)^2}\cdot\frac{x^2-3x-18}{x^2-2x-24} \Rightarrow \frac{(x+4)(x-4)}{(x-4)(x-4)}\cdot\frac{(x-6)(x+3)}{(x-6)(x+4)} \Rightarrow \frac{(x+4)(x-4)(x-6)(x+3)}{(x+4)(x-4)(x-6)(x-4)} \Rightarrow \frac{x+3}{x-4}$

b. $\frac{x^2-1}{x^2-6x-7} \div \frac{x^3+x^2-2x}{x-7} \Rightarrow \frac{(x+1)(x-1)}{(x-7)(x+1)}\cdot\frac{(x-7)}{x(x+2)(x-1)} \Rightarrow \frac{(x+1)(x-1)(x-7)}{(x+1)(x-1)(x-7)x(x+2)} \Rightarrow \frac{1}{x(x+2)}$

Here are some more to try. Multiply or divide each pair of rational expressions. Simplify the result. Assume the denominator is not equal to zero.

1. $\frac{x^2+5x+6}{x^2-4x}\cdot\frac{4x}{x+2}$
2. $\frac{x^2-2x}{x^2-4x+4}\div\frac{4x^2}{x-2}$
3. $\frac{x^2-16}{(x-4)^2}\cdot\frac{x^2-3x-18}{x^2-2x-24}$
4. $\frac{x^2-x-6}{x^2+3x-10}\cdot\frac{x^2+2x-15}{x^2-6x+9}$
5. $\frac{x^2-x-6}{x^2-x-20}\cdot\frac{x^2+6x+8}{x^2-x-6}$
6. $\frac{x^2-x-30}{x^2+13x+40}\cdot\frac{x^2+11x+24}{x^2-9x+18}$
7. $\frac{15-5x}{x^2-x-6}\div\frac{5x}{x^2+6x+8}$
8. $\frac{17x+119}{x^2+5x-14}\div\frac{9x-1}{x^2-3x+2}$
9. $\frac{2x^2-5x-3}{3x^2-10x+3}\cdot\frac{9x^2-1}{4x^2+4x+1}$
10. $\frac{x^2-1}{x^2-6x-7}\div\frac{x^3+x^2-2x}{x-7}$
11. $\frac{3x-21}{x^2-49}\div\frac{3x}{x^2+7x}$
12. $\frac{x^2-y^2}{x+y}\cdot\frac{1}{x-y}$
13. $\frac{y^2-y}{w^2-y^2}\div\frac{y^2-2y+1}{1-y}$
14. $\frac{y^2-y-12}{y+2}\div\frac{y-4}{y^2-4y-12}$
15. $\frac{x^2+7x+10}{x+2}\div\frac{x^2+2x-15}{x+2}$

Answers:

1. $\frac{4(x+3)}{x-4}$
2. $\frac{1}{4x}$
3. $\frac{x+3}{x-4}$
4. $\frac{x+2}{x-2}$
5. $\frac{x+2}{x-5}$
6. $\frac{x+3}{x-3}$
7. $\frac{-x-4}{x}$
8. $\frac{17(x-1)}{9x-1}$
9. $\frac{3x+1}{2x+1}$
10. $\frac{1}{x(x+2)}$
11. 1
12. 1
13. $\frac{-y}{w^2-y^2}$
14. $(y+3)(y-6)$
15. $\frac{x+2}{x-3}$

Checkpoint 6B

Problem 6-145

Adding and Subtracting Rational Expressions

Answers to problem 6-145: a: $\frac{2(x+1)}{x+3}$, b: $\frac{3x^2-5x-3}{(2x+1)^2}$

Addition and subtraction of rational expressions uses the same process as simple numerical fractions. First, if necessary find a common denominator. Second, convert the original fractions to equivalent ones with the common denominator. Third, add or subtract the new numerators over the common denominator. Finally, factor the numerator and denominator and simplify, if possible. Note that these steps are only valid provided that the denominator is not zero.

Example 1: $\frac{3}{2(n+2)}+\frac{3}{n(n+2)}$

Solution:

The least common multiple of $2(n+2)$ and $n(n+2)$ is $2n(n+2)$.

$$\frac{3}{2(n+2)}+\frac{3}{n(n+2)}$$

To get a common denominator in the first fraction, multiply the fraction by $\frac{n}{n}$, a form of the number 1. Multiply the second fraction by $\frac{2}{2}$.

$$=\frac{3}{2(n+2)}\cdot\frac{n}{n}+\frac{3}{n(n+2)}\cdot\frac{2}{2}$$

Multiply the numerator and denominator of each term. It may be necessary to distribute the numerator.

$$=\frac{3n}{2n(n+2)}+\frac{6}{2n(n+2)}$$

Add, factor, and simplify the result. (Note: $n \neq 0$ *or* -2)

$$=\frac{3n+6}{2n(n+2)}\Rightarrow\frac{3(n+2)}{2n(n+2)}\Rightarrow\frac{3}{2n}$$

Example 2: $\frac{2-x}{x+4}+\frac{3x+6}{x+4}$

Solution:

$$\frac{2-x}{x+4}+\frac{3x+6}{x+4}\Rightarrow\frac{2-x+3x+6}{x+4}\Rightarrow\frac{2x+8}{x+4}\Rightarrow\frac{2(x+4)}{x+4}\Rightarrow 2$$

Example 3: $\frac{3}{x-1}-\frac{2}{x-2}$

Solution:

$$\frac{3}{x-1}-\frac{2}{x-2}\Rightarrow\frac{3}{x-1}\cdot\frac{x-2}{x-2}-\frac{2}{x-2}\cdot\frac{x-1}{x-1}\Rightarrow\frac{3x-6-2x+2}{(x-1)(x-2)}\Rightarrow\frac{x-4}{(x-1)(x-2)}$$

Now we can go back and solve the original problems.

a. $\frac{4}{x^2+5x+6}+\frac{2x}{x+2} \Rightarrow \frac{4}{(x+3)(x+2)}+\frac{2x}{(x+2)}\cdot\frac{(x+3)}{(x+3)} \Rightarrow \frac{2x^2+6x+4}{(x+2)(x+3)} \Rightarrow \frac{2(x+2)(x+1)}{(x+2)(x+3)} \Rightarrow \frac{2(x+1)}{(x+3)}$

b. $\frac{3x^2+x}{(2x+1)^2}-\frac{3}{(2x+1)} \Rightarrow \frac{3x^2+x}{(2x+1)^2}-\frac{3}{(2x+1)}\cdot\frac{(2x+1)}{(2x+1)} \Rightarrow \frac{3x^2+x-6x-3}{(2x+1)^2} \Rightarrow \frac{3x^2-5x-3}{(2x+1)^2}$

Here are a few more to try. Add or subtract each expression and simplify the result. In each case assume the denominator does not equal zero.

1. $\frac{x}{(x+2)(x+3)}+\frac{2}{(x+2)(x+3)}$
2. $\frac{x}{x^2+6x+8}+\frac{4}{x^2+6x+8}$
3. $\frac{b^2}{b^2+2b-3}+\frac{-9}{b^2+2b-3}$
4. $\frac{2a}{a^2+2a+1}+\frac{2}{a^2+2a+1}$
5. $\frac{x+10}{x+2}+\frac{x-6}{x+2}$
6. $\frac{a+2b}{a+b}+\frac{2a+b}{a+b}$
7. $\frac{3x-4}{3x+3}-\frac{2x-5}{3x+3}$
8. $\frac{3x}{4x-12}-\frac{9}{4x-12}$
9. $\frac{6a}{5a^2+a}-\frac{a-1}{5a^2+a}$
10. $\frac{x^2+3x-5}{10}-\frac{x^2-2x+10}{10}$
11. $\frac{6}{x(x+3)}+\frac{2x}{x(x+3)}$
12. $\frac{5}{x-7}+\frac{3}{4(x-7)}$
13. $\frac{5x+6}{x^2}-\frac{5}{x}$
14. $\frac{2}{x+4}-\frac{x-4}{x^2-16}$
15. $\frac{10a}{a^2+6a}-\frac{3}{3a+18}$
16. $\frac{3x}{2x^2-8x}+\frac{2}{x-4}$
17. $\frac{5x+9}{x^2-2x-3}+\frac{6}{x^2-7x+12}$
18. $\frac{x+4}{x^2-3x-28}-\frac{x-5}{x^2+2x-35}$
19. $\frac{3x+1}{x^2-16}-\frac{3x+5}{x^2+8x+16}$
20. $\frac{7x-1}{x^2-2x-3}-\frac{6x}{x^2-x-2}$

Answers:

1. $\frac{1}{x+3}$
2. $\frac{1}{x+2}$
3. $\frac{b-3}{b-1}$
4. $\frac{2}{a+1}$
5. 2
6. 3
7. $\frac{1}{3}$
8. $\frac{3}{4}$
9. $\frac{1}{a}$
10. $\frac{x-3}{2}$
11. $\frac{2}{x}$
12. $\frac{23}{4(x-7)}=\frac{23}{4x-28}$
13. $\frac{6}{x^2}$
14. $\frac{1}{x+4}$
15. $\frac{9}{a+6}$
16. $\frac{7}{2(x-4)}=\frac{7}{2x-8}$
17. $\frac{5(x+2)}{(x-4)(x+1)}=\frac{5x+10}{x^2-3x-4}$
18. $\frac{14}{(x-7)(x+7)}=\frac{14}{x^2-49}$
19. $\frac{4(5x+6)}{(x-4)(x+4)^2}$
20. $\frac{x+2}{(x-3)(x-2)}=\frac{x+2}{x^2-5x+6}$

Checkpoint 7A

Problem 7-67

Finding the x- and y-Intercepts of a Quadratic Function

Answers to problem 7-67: y-intercept: $(0,-17)$, x-intercepts: $(-2+\sqrt{21},0)$ and $(-2-\sqrt{21},0)$

The y-intercepts of an equation are the points at which the graph crosses the y-axis. To find the y-intercept of an equation, substitute $x=0$ into the equation and solve for y.

The x-intercepts of an equation are the points at which the graph crosses the x-axis. To find the x-intercepts of an equation, substitute $y=0$ into the equation and solve for x. For a quadratic, you can do this by factoring and using the Zero Product Property or by using the Quadratic Formula, as well as other methods.

Example 1: Find the x-intercepts of the graph of the equation $y=x^2+4x-12$.

Solution: If $y=0$, then: $0=x^2+4x-12$

By factoring and using the Zero Product Property: $0=(x+6)(x-2)$

$x+6=0$ or $x-2=0$

$x=-6$ or $x=2$

Answers: The x-intercepts are $(-6,0)$ and $(2,0)$.

Example 2: Find the x-intercepts of the graph of the equation $y=2x^2-3x-3$.

Solution: If $y=0$, then: $0=2x^2-3x-3$

Since we cannot factor the trinomial we use the Quadratic Formula to solve for x.

If $ax^2+bx+c=0$ then: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Substitute $a=2$, $b=-3$, $c=-3$. $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-3)}}{2(2)}$

Simplify. $x=\frac{3\pm\sqrt{9+24}}{4}=\frac{3\pm\sqrt{33}}{4}$

Find $\sqrt{33}$ approximately: $\approx\frac{3\pm5.745}{4}$, so $\frac{3+5.745}{4}$ and $\frac{3-5.745}{4}$.

Answers: Simplify the fractions and the x-intercepts are approximately $(2.19,0)$ and $(-0.69,0)$. They can be expressed in exact form as $\left(\frac{3+\sqrt{33}}{4},0\right)$ and $\left(\frac{3-\sqrt{33}}{4},0\right)$.

Now we can find the x- and y-intercepts in the original problem.

Find the x- and y-intercepts of the graph of $y = x^2 + 4x - 17$.

Solution: To find the y-intercept, let $x = 0$ so $y = (0)^2 + 4(0) - 17 = -17$.
To find the x-intercept, let $y = 0$ so $0 = x^2 + 4x - 17$.
Since we cannot factor we use the Quadratic Formula with $a = 1$, $b = 4$ and $c = -17$.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-17)}}{2(1)} = \frac{-4 \pm \sqrt{16+68}}{2} = \frac{-4 \pm \sqrt{84}}{2} = \frac{-4 \pm 2\sqrt{21}}{2} = -2 \pm \sqrt{21}$$

Answers: The y-intercept is $(0, -17)$. The x-intercepts are $(-2 \pm \sqrt{21}, 0)$, or approximately $(2.58, 0)$ and $(-6.58, 0)$.

Here are some more to try. Find the x- and y-intercepts for the graphs of each equation.

1. $y = x^2 + 2x - 15$
2. $y = 2x^2 + 7x + 3$
3. $y = 3x^2 - 5x + 2$
4. $y = 4x^2 - 8x$
5. $y = 2x^2 - 9x - 35$
6. $y = 2x^2 - 11x + 5$
7. $3x^2 + 2 + 7x = y$
8. $8x^2 + 10x + 3 = y$
9. $y + 2 = x^2 - 5x$
10. $(x-3)(x+4) - 7x = y$
11. $-4x^2 + 8x + 3 = y$
12. $0.009x^2 - 0.86x + 2 = y$
13. $y = 2x^3 - 50x$
14. $y = 3x^2 + 4x$

Answers:

1. $(3,0), (-5,0)$ and $(0,-15)$
2. $\left(-\frac{1}{2},0\right), (-3,0)$ and $(0,3)$
3. $\left(\frac{2}{3},0\right), (1,0)$ and $(0,2)$
4. $(0,0), (2,0)$ and $(0,0)$
5. $(7,0), \left(-\frac{5}{2},0\right)$, and $(0,-35)$
6. $(5,0), \left(\frac{1}{2},0\right)$, and $(0,5)$
7. $\left(-\frac{1}{3},0\right), (-2,0)$, and $(0,2)$
8. $\left(-\frac{3}{4},0\right), \left(-\frac{1}{2},0\right)$, and $(0,3)$
9. $\left(\frac{5 \pm \sqrt{33}}{2},0\right)$ or $(\approx 5.37,0), (\approx -0.37,0)$, and $(0,-2)$
10. $\left(\frac{-6 \pm \sqrt{84}}{2},0\right)$ or $(\approx 7.58,0), (\approx -1.58,0)$, and $(0,-12)$
11. $\left(\frac{-8 \pm \sqrt{112}}{-8},0\right)$ or $(\approx -0.32,0), (\approx 2.32,0)$, and $(0,3)$
12. $\left(\frac{0.86 \pm \sqrt{0.6676}}{0.018},0\right)$ or $(\approx 2.39,0), (\approx 93.17,0)$, and $(0,2)$
13. $(0,0), (5,0), (-5,0)$, and $(0,0)$
14. $(0,0), \left(-\frac{4}{3},0\right)$, and $(0,0)$

Checkpoint 7B

Problem 7-131

Completing the Square to Find the Vertex of a Parabola

Answers to problem 7-131: Graphing form: $y = 2(x-1)^2 + 3$, vertex $(1, 3)$
See graph in solution that follows examples.

If a quadratic function is in graphing form then the vertex can be found easily and a sketch of the graph can be made quickly. If the equation of the parabola is not in graphing form, the equation can be rewritten in graphing form by completing the square.

First, recall that $y = x^2$ is the parent equation for quadratic functions and the general equation can be written in graphing form as $y = a(x-h)^2 + k$ where (h, k) is the vertex, and relative to the parent graph the function has been:

- Vertically stretched, if the absolute value of a is greater than 1
- Vertically compressed, if the absolute value of a is less than 1
- Reflected across the x-axis, if a is less than 0.

Example 1: Complete the square to change $y = x^2 + 8x + 10$ into graphing form and name the vertex.

Solution: Use an area model to make $x^2 + 8x$ into a perfect square. To do this, use half of the coefficient of the x-term on each side of the area model, and complete the upper right corner of the square, as shown at right.

4	$4x$	16
x	x^2	$4x$
	x	4

Your square shows that $(x+4)^2 = x^2 + 8x + 16$. Your original expression is $x^2 + 8x + 10$, which is 6 fewer than $(x+4)^2$. So you can write $y = x^2 + 8x + 10 = (x+4)^2 - 6$.

Because the function is now in graphing form, $y = a(x-h)^2 + k$, you know the vertex is (h, k), in this case $(-4, -6)$.

Example 2: Complete the square to change $y = x^2 + 5x + 7$ into graphing form and name the vertex.

Solution: We need to make $x^2 + 5x$ into a perfect square. Again, we take half of the coefficient of x and fill in the upper right of the area model, as shown below.

2.5	$2.5x$	6.25
x	x^2	$2.5x$
	x	2.5

The area model shows that $(x+2.5)^2 = x^2 + 5x + 6.25$. Your original expression, $y = x^2 + 5x + 7$, is 0.75 more than $(x+2.5)^2$.
So you can write: $y = x^2 + 5x + 7 = (x+2.5)^2 + 0.75$ and the vertex is $(-2.5, 0.75)$.

Example 3: Complete the square to change $y = 2x^2 - 6x + 2$ into graphing form and name the vertex.

Solution: This problem is different because the x^2 term has a coefficient. First factor the 2 out of the quadratic expression so that an x^2 term remains, as follows: $y = 2x^2 - 6x + 2 = 2(x^2 - 3x + 1)$. Then make $x^2 - 3x$ into a perfect square as before:

-1.5	$-1.5x$	2.25
x	x^2	$-1.5x$
	x	-1.5

Since $(x-1.5)^2 = x^2 - 3x + 2.25$, the original expression $x^2 - 3x + 1$ is 1.25 less than $(x-1.5)^2$. You can write:

$y = 2x^2 - 6x + 2$

$y = 2(x^2 - 3x + 1)$

$y = 2\left((x-1.5)^2 - 1.25\right)$

which can be distributed to give

$y = 2(x-1.5)^2 - 2.5$

The vertex is $(1.5, -2.5)$.

Now we can go back and solve the original problem.

Complete the square to change the equation $y = 2x^2 - 4x + 5$ to graphing form, identify the vertex of the parabola, and sketch its graph.

Solution: Factor out the coefficient of x^2, resulting in $y = 2\left(x^2 - 2x + \frac{5}{2}\right)$. Make a perfect square out of $x^2 - 2x$:

-1	$-1x$	1
x	x^2	$-1x$
	x	-1

Since $(x-1)^2 = x^2 - 2x + 1$, the original expression $x^2 - 2x + \frac{5}{2}$ is $\frac{3}{2}$ more than $(x-1)^2$. You can write:

$y = 2x^2 - 4x + 5$

$y = 2(x^2 - 2x + \frac{5}{2})$

$y = 2\left((x-1)^2 + \frac{3}{2}\right)$

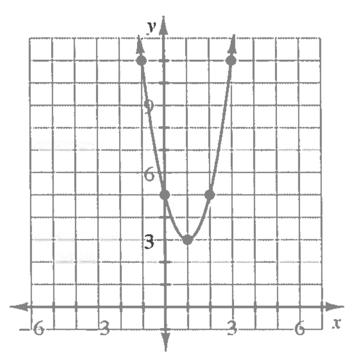
which can be distributed to give

$y = 2(x-1)^2 + 3$

Because the function is now in graphing form, $y = a(x-h)^2 + k$, you know the vertex is $(h, k) = (1, 3)$.

To sketch the graph, we start by plotting the vertex, $(1, 3)$. From the standard form, $y = 2x^2 - 4x + 5$, we see that the y-intercept is $(0, 5)$, because when $x = 0$, $y = 5$. By symmetry, $(2, 5)$ must also be a point. Thus we get the graph at the right. If desired, additional points can be found by recognizing that the shape of this parabola is the shape of $y = x^2$ stretched by a factor of 2.

Answer: Graphing form: $y = 2(x-1)^2 + 3$;
Vertex: $(1, 3)$;
See graph at right.



Here are some more to try. Write each equation in graphing form. If needed, complete the square to do so. Then state the vertex, y-intercept, and the stretch factor and sketch a graph.

1. $y=x^2-6x+9$
2. $y=x^2+3$
3. $y=x^2-4x$
4. $y=x^2+2x-3$
5. $y=x^2+5x+1$
6. $y=x^2-\frac{1}{3}x$
7. $y=3x^2-6x+1$
8. $y=5x^2+20x-16$
9. $y=-x^2-6x+10$

Answers:

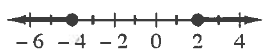
1. $y=(x-3)^2$; $(3,0)$; $(0,9)$; $a=1$
2. $y=(x-0)^2+3$; $(0,3)$; $(0,3)$; $a=1$
3. $y=(x-2)^2-4$; $(2,-4)$; $(0,0)$; $a=1$
4. $y=(x+1)^2-4$; $(-1,-4)$; $(0,-3)$ $a=1$
5. $y=\left(x+\frac{5}{2}\right)^2-5\frac{1}{4}$; $\left(-\frac{5}{2},-5\frac{1}{4}\right)$; $(0,1)$; $a=1$
6. $y=\left(x-\frac{1}{6}\right)^2-\frac{1}{36}$; $\left(\frac{1}{6},-\frac{1}{36}\right)$; $(0,0)$; $a=1$
7. $y=3(x-1)^2-2$; $(1,-2)$; $(0,1)$; $a=3$
8. $y=5(x+2)^2-36$; $(-2,-36)$; $(0,-16)$; $a=5$
9. $y=-(x+3)^2+19$ $(-3,19)$; $(0,10)$; $a=-1$

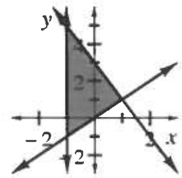
Checkpoint 8A

Problem 8-127

Solving and Graphing Inequalities

Answers to problem 8-127:

a. 

b. 

There are several methods for graphing different types of inequalities but there is one way that works for all types: Solve as you would an equation then use the solutions to break the graph into regions. If testing a point from a region makes the inequality "true," shade in that region as it is part of the solution. If a point from the region makes the inequality "false," then that region is not part of the solution.

Example 1: Graph $x^2 + 5x + 4 < 0$.

Solution: Change the inequality into an equation and solve.

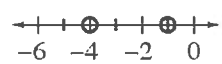
$$x^2 + 5x + 4 = 0$$
$$(x+4)(x+1) = 0$$
$$x = -4 \text{ or } x = -1$$

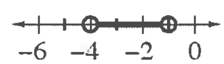
Place the solutions on a number line to break the line into three regions. Use open circles since the original problem is a strict inequality. Test one point from each region in the original inequality. For example testing $x = -5$ we find:

$$(-5)^2 + 5(-5) + 4 < 0$$
$$4 < 0 \text{ False}$$

So that region is not shaded in. Continue in the same manner with one point from each of the other regions.

test $x=-5$	test $x=-2$	test $x=0$
F	T	F



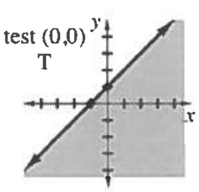


The solution may also be written $-4 < x < -1$.

Example 2: Graph the system $y \le x + 1$
$y \ge x^2 - 4$

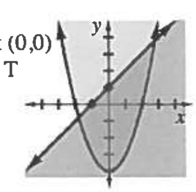
Solution: Graph the line $y = x + 1$ and check the point $(0, 0)$. The point makes the first inequality "true" so the shading is in the $(0, 0)$ region or below the line.

test (0,0)
T



Next graph the parabola $y = x^2 - 4$ and again finding the point $(0, 0)$ to be "true" in the inequality, the shading is inside the parabola. The overlapping shaded region is the solution to the system.

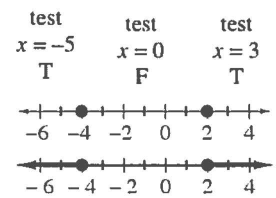
test (0,0)
T



Now we can go back and solve the original problems.

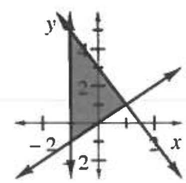
a. $|x+1| \geq 3$

Solution: If $|x+1| = 3$ then $x+1 = \pm 3$ or $x = 2, -4$. Using solid dots to divide the number line into three regions and testing a point from each region gives the answer graph at right. This can also be written as: $x \leq -4$ or $x \geq 2$.



b. Graph: $y \leq -2x+3 \quad y \geq x \quad x \geq -1$

Solution: Start by looking at the equation of the line that marks the edge of each inequality. The first has slope -2 and y-intercept $(0, 3)$. Checking $(0, 1)$ gives a true statement so we shade below the solid line. The second line has slope 1 and y-intercept $(0, 0)$. Again checking $(0, 1)$ gives a true statement, so we shade above the solid line. The third is a vertical line at $x = -1$. Checking a point tells us to shade the right side. The overlapping shading is a triangle with vertices $(-1, 5)$, $(1, 1)$, and $(-1, -1)$. See the answer graph at right.



Here are a few more to try. In problems 1 through 6, graph each inequality.
In problems 7 through14 graph the solution region for each system of inequalities.

1. $|2x+3| \leq 7$

2. $x^2 - 2x - 3 > 0$

3. $4 - x^2 \leq 0$

4. $|4r - 2| > 8$

5. $3m^2 \leq 9m$

6. $-|x+3| < 10$

7. $y \leq -x+2$
   $y \leq 3x - 6$

8. $y \geq \frac{2}{3}x + 4$
   $y \leq \frac{7}{12}x + 5$

9. $x < 3$
   $y \geq -2$

10. $y \leq 4x + 16$
    $y \geq -\frac{4}{3}x - 4$

11. $y \geq x^2 - 4$
    $y < -3x + 1$

12. $y < 2x + 5$
    $y \geq |x+1|$

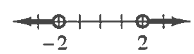
13. $y < x^2 - 2x - 3$
    $y \leq \frac{1}{2}x + 3$

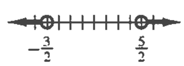
14. $y \leq -\frac{1}{2}x + 2$
    $y \geq (x+1)^2 - 3$

Answers:

1. 

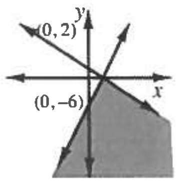
2. 

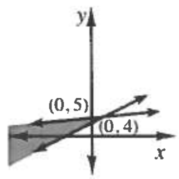
3. 

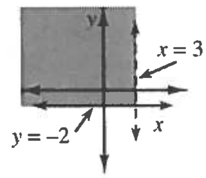
4. 

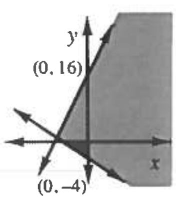
5. 

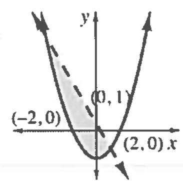
6. 

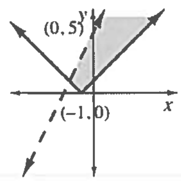
7. 

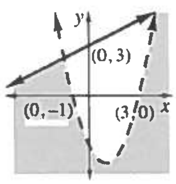
8. 

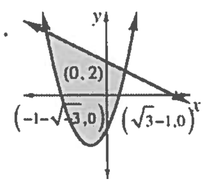
9. 

10. 

11. 

12. 

13. 

14. 

Checkpoint 8B

Problem 8-174

Solving Complicated Equations

Answers to problem 8-174: a: $x = 5$ or 1, b: $x = 4$ or 0, c: $x = 7$, d: $x = 1$

Often the best way to solve a complicated equation is to use a method such as the Looking Inside or Undoing Methods. Checking your answer(s) is important because sometimes solutions are accurately found but will not work in the original equation. These answers are called extraneous solutions.

Example 1: Solve $3(x+1)^3 - 1 = 80$

Solution: In this case we will undo everything.

		check
$3(x+1)^3 - 1 = 80$	original problem	
$3(x+1)^3 = 81$	add 1 to undo -1	$3(2+1)^3 = 81$
$(x+1)^3 = 27$	division undoes multiplication	$3(3)^3 = 81$
$x+1 = 3$	cube root undoes cubing	$3(27) = 81$
$x = 2$	subtract 1 to undo $+1$	$81 = 81$

Example 2: Solve $|2x+5| = x+4$

Solution: In this case we will use the Look Inside Method and the fact that absolute value of a quantity and its opposite are the same.

			checks	
$2x+5 = x+4$	or	$-(2x+5) = x+4$		
$x = -1$		$-2x-5 = x+4$	$\lvert 2(-1)+5 \rvert = -1+4$	$\lvert 2(-3)+5 \rvert = -3+4$
		$-3x = 9$	$\lvert 3 \rvert = \lvert 3 \rvert$	$\lvert -1 \rvert = \lvert 1 \rvert$
		$x = -3$	$3 = 3$ True	$1=1$ True

The solutions are $x = -1$, $x = -3$.

Now we can go back and solve the original problems.

a.

$$2|x-3|+7=11$$
$$2|x-3|=4$$
$$|x-3|=2$$
$$x-3=2 \text{ or } -(x-3)=2$$
$$x=5 \text{ or } x=1$$

checks

$$2|5-3|+7=11 \qquad 2|1-3|+7=11$$
$$2|2|+7=11 \qquad 2|-2|+7=11$$
$$4+7=11 \text{ True} \qquad 4+7=11 \text{ True}$$

The solutions are $x=5, x=1$.

b.

$$4(x-2)^2=16$$
$$(x-2)^2=4$$
$$x-2=\pm 2$$
$$x=2\pm 2=4 \text{ or } 0$$

checks

$$4(4-2)^2=16 \qquad 4(0-2)^2=16$$
$$4(2)^2=16 \qquad 4(-2)^2=16$$
$$4(4)=16 \text{ True} \qquad 4(4)=16 \text{ True}$$

The solutions are $x=4, x=0$.

c.

$$\sqrt{x+18}=x-2$$
$$(\sqrt{x+18})^2=(x-2)^2$$
$$x+18=x^2-4x+4$$
$$0=x^2-5x-14$$
$$0=(x-7)(x+2)$$
$$x=7 \text{ or } x=-2$$

checks

$$\sqrt{7+18}=7-2 \qquad \sqrt{-2+18}=-2-2$$
$$\sqrt{25}=5 \text{ True} \qquad \sqrt{16}=-4 \text{ False}$$

The only solution is $x=7$.

d.

$$|2x+5|=3x+4$$
$$2x+5=3x+4 \text{ or } -(2x+5)=3x+4$$
$$x=1 \qquad \text{or} \qquad x=-\tfrac{9}{5}$$

The only solution is $x=1$.

checks

$$|2(1)+5|=3(1)+4 \qquad \left|2(-\tfrac{9}{5})+5\right|=3(-\tfrac{9}{5})+4$$
$$|7|=7 \text{ True} \qquad \left|-\tfrac{18}{5}+\tfrac{25}{5}\right|=-\tfrac{27}{5}+\tfrac{20}{5}$$
$$\left|\tfrac{7}{5}\right|=-\tfrac{7}{5} \text{ False}$$

Here are some more to try. Solve each equation and check your solutions.

1. $|x+4|+3=17$
2. $|3w-6|-2=19$
3. $\sqrt{3w+4}-2=2$
4. $\sqrt{3x+13}=x+5$
5. $\sqrt[3]{x}-2=5$
6. $(x+1)^2+1=6$
7. $|2y-4|=12$
8. $|3m+5|=5m+2$
9. $\sqrt{y+7}+5=y$
10. $\sqrt{5-m}=m+1$
11. $\frac{2(x-1)^2}{3}-7=1$
12. $3(x-2)^3=81$
13. $|2m+5|=m+4$
14. $|2y+8|=3y+7$
15. $\sqrt{x+7}-x=1$
16. $\sqrt{y+2}=y$
17. $\sqrt[4]{x+1}+2=5$
18. $\frac{1}{2}(x+5)^3+1=10$
19. $\sqrt{x}+2=x$
20. $\sqrt{x}+2=\sqrt{x+6}$

Answers:

1. $x=-18, 10$
2. $w=-5, 9$
3. $w=4$
4. $x=-3, -4$
5. $x=343$
6. $x=-1\pm\sqrt{5}$
7. $y=-4, 8$
8. $m=\frac{3}{2}$
9. $y=9$
10. $m=1$
11. $x=1\pm\sqrt{12}$
12. $x=5$
13. $m=-1, -3$
14. $y=1$
15. $x=2$
16. $y=2$
17. $x=80$
18. $x=-5+\sqrt[3]{18}$
19. $x=4$
20. $x=\frac{1}{4}$

Checkpoint 9A

Problem 9-41

Writing and Solving Exponential Equations

Answers to problem 9-41:

a. The more rabbits you have, the more new ones you get, a linear model would grow by the same number each year. A sine function would be better if the population rises and falls, but more data would be needed to apply this model.

b. $R = 80,000(5.4772...)^t$ c. ≈ 394 million

d. 1859, it seems okay that they grew to 80,000 in 7 years, *if* they are growing exponentially.

e. No, since it would predict a huge number of rabbits now. The population probably leveled off at some point or dropped drastically and rebuilt periodically.

Exponential functions are equations of the form $y = ab^x + c$ where a represents the initial value, b represents the multiplier, c represents the horizontal asymptote, and x often represents the time. Some problems simply involve substituting in the given information into the equation and then doing calculations. If you are trying to solve for the time (x), then you will usually need to use logarithms. If you need to find the multiplier (m), then you will need to find a root. Note that we often assume that $c = 0$, unless we are told otherwise.

Example 1: Lunch at our favorite fast food stands cost $6.50. The price has steadily increased 4% per year for many years.

Question 1: What will lunch cost in 10 years?

Solution: In this case, we can use $6.50 as the initial value and the multiplier is 1.04, so the equation for the situation is $y = 6.50(1.04)^x$. The time we are interested in here is 10 years. Substituting into the equation, $y = 6.50(1.04)^{10} = \$9.62$.

Question 2: What did it cost 10 years ago?

Solution: Using the same equation, only using –10 for the years, $y = 6.50(1.04)^{-10} = \$4.39$.

Question 3: How long before lunch costs $10?

Solution: Again, we use the same equation, but this time we know the y-value, but not the value of x. To solve for x, we use logarithms, as shown in the work at right.

$$\$10 = 6.50(1.04)^x$$
$$1.04^x = \frac{10}{6.5}$$
$$\log 1.04^x = \log\left(\frac{10}{6.5}\right)$$
$$x \log 1.04 = \log\left(\frac{10}{6.5}\right)$$
$$x = \frac{\log\left(\frac{10}{6.5}\right)}{\log 1.04} \approx 11$$

Answer: About 11 years.

Example 2: Tickets for a big concert first went on sale three weeks ago for $60. This week people are charging $100. Write an equation that represents the cost of the tickets *w* weeks from the time that they went on sale. Assume that they continue to increase in the same way.

Solution: To find the multiplier, we can use what we are given. The initial value is $60, the time is 3 weeks, and the final value is $100. This gives $100 = 60k^3$. Solving for k gives $k = \sqrt[3]{\frac{100}{60}} \approx 1.186$.

Answer: The equation is approximately $y = 60(1.186)^w$.

Now we can go back and solve parts (b), (c), and (d) of the original problem.

When rabbits were first brought to Australia, they had no natural enemies. There were about 80,000 rabbits in 1866. Two years later, in 1868, the population had grown to over 2,400,000!

b. Write an exponential equation for the number of rabbits t years after 1866.

Solution: For 1866, 80000 would be the initial value, time would be 2 years, and the final amount would be 2400000, which gives the equation $2400000 = 80000m^2$. Solving for m, we get $30 = m^2$ so $m = \sqrt{30} \approx 5.477$. Thus the equation is approximately $R = 80{,}000(5.477)^t$.

c. How many rabbits do you predict would have been present in 1871?

Solution: The initial value is still 80,000, the multiplier ≈ 5.477, and now the time is 5 years. This gives $80{,}000(5.477)^5 \approx$ 394 million.

d. According to your model, in what year was the first pair of rabbits introduced into Australia?

Solution: In this situation, we use 2 as the initial value, 80000 as the final value, and the multiplier is still 5.477 but now the time is not known. Using these values, we get $80000 = 2(5.477)^x$, which is solved at right. The answer 6.23 tells us that approximately 6.23 years had passed between the time of the first pair of rabbits, and the time when there were 80000. Thus, rabbits would have been introduced sometime in 1859.

$$80000 = 2(5.477)^x$$
$$40000 = (5.477)^x$$
$$\log(5.477)^x = \log 40000$$
$$x\log(5.477) = \log 40000$$
$$x = \frac{\log(40000)}{\log(5.477)} \approx 6.23$$

Here are some more to try:

1. A DVD loses 60% of its value every year it is in the store. The DVD costs \$80 new. Write a function that represents its value in t years. What is it worth after 4 years?

2. Inflation is at a rate of 7% per year. Evan's favorite bread now costs \$1.79. What did it cost 10 years ago? How long before the cost of the bread doubles?

3. A bond that appreciates 4% per year will be worth \$146 in five years. Find the current value.

4. Sixty years ago, when Sam's grandfather was a kid, he could buy his friend dinner for \$1.50. If that same dinner now costs \$25.25 and inflation was consistent, write an equation that models the cost for the dinner at different times.

5. A two-bedroom house in Omaha is now worth \$110,000. If it appreciates at a rate of 2.5% per year, how long will it take to be worth \$200,000?

6. A car valued at \$14,000 depreciates 18% per year. After how many years will the value have depreciated to \$1000?

Answers:

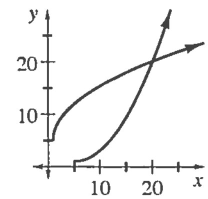
1. $y = 80(0.4)^t$, \$2.05
2. \$0.91, 10.2 years
3. \$120
4. $y = 1.50(1.048)^x$
5. 24.2 years
6. 13.3 years

Checkpoint 9B

Problem 9-111

Finding the Equation for the Inverse of a Function

Answers to problem 9-111:



a. $f^{-1}(x) = \frac{1}{3}\left(\frac{x-5}{2}\right)^2 + 1 = \frac{1}{12}(x-5)^2 + 1$ for $x \geq 5$

b. See graph at right.

To find the equation for the inverse of a function, you can interchange the x and y variables and then solve for y. This also means that the coordinates of points that are on the graph of the function will be reversed on the graph of the inverse. Here are some examples:

Example 1: Write the equation for the inverse of $y = 2(x+3)$.

Solution: We can interchange the x and the y to get the equation of the inverse. To get our final answer, we solve for y, as shown at right.

$$2(y+3) = x$$
$$(y+3) = \frac{x}{2}$$
$$y = \frac{x}{2} - 3$$

Answer: $y = \frac{x}{2} - 3$

Example 2: Write the equation for the inverse of $y = \frac{1}{2}(x+4)^2 + 1$.

Solution: Again, we can interchange the x and the y to get the equation of the inverse and then solve for y to get our answer in $y =$ form, as shown at right.

$$x = \frac{1}{2}(y+4)^2 + 1$$
$$\frac{1}{2}(y+4)^2 = x - 1$$
$$(y+4)^2 = 2x - 2$$
$$y + 4 = \pm\sqrt{2x-2}$$
$$y = -4 \pm \sqrt{2x-2}$$

Answer: $y = -4 \pm \sqrt{2x-2}$. Note that because of the $\pm$, this inverse is not a function.

Example 3: Write the equation for the inverse of $y = -\frac{2}{3}x + 6$.

Solution: Interchanging the x and the y, we get $x = -\frac{2}{3}y + 6$. Solving for y gives $y = -\frac{3}{2}(x-6) = -\frac{3}{2}x + 9$.

Answer: $y = -\frac{3}{2}x + 9$

Example 4: Write the equation for the inverse of $y = \sqrt{x-2} + 3$.

Solution: Again, we exchange x and y and then solve for y, as shown at right.

$$x = \sqrt{y-2} + 3$$
$$\sqrt{y-2} = x - 3$$
$$y - 2 = (x-3)^2$$
$$y = (x-3)^2 + 2$$

The original function is half of a sleeping parabola, so the inverse is only half of a parabola as well. Thus the domain of the inverse is restricted to $x \geq 3$.

Answer: $y = (x-3)^2 + 2$ in the domain $x \geq 3$.

Now we can go back and solve the original problem.

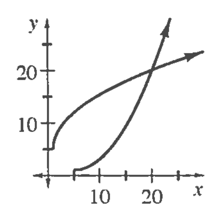
Find the equation for the inverse of the function $y = 2\sqrt{3(x-1)} + 5$. Then sketch the graph of both the original and the inverse.

Solution: Interchanging x and y we get $x = 2\sqrt{3(y-1)} + 5$. We then solve for y, as shown at right. This equation is simplified to get $y = \frac{(x-5)^2}{12} + 1$.

$$x = 2\sqrt{3(y-1)} + 5$$
$$2\sqrt{3(y-1)} = x - 5$$
$$\sqrt{3(y-1)} = \tfrac{x-5}{2}$$
$$3(y-1) = \left(\tfrac{x-5}{2}\right)^2$$
$$y - 1 = \tfrac{1}{3}\left(\tfrac{x-5}{2}\right)^2$$
$$y = \tfrac{1}{3}\left(\tfrac{x-5}{2}\right)^2 + 1$$

Note that the domain and range of the inverse are the interchanged domain and range of the original function. In other words, the original function has a domain of $x \geq 1$ and range of $y \geq 5$. The domain of the inverse, then, is $x \geq 5$ and the range is $y \geq 1$.

As you can see by the graph at right, the points on the inverse graph, have interchanged coordinates from the points on the graph of the original function. For example, two points on the original graph are $(1, 5)$ and $(4, 11)$. The corresponding points on the graph of the inverse are $(5, 1)$ and $(11, 4)$.



Answer: $y = \frac{(x-5)^2}{12} + 1$ in the domain $x \geq 5$.

Here are some more to try. Find an equation for the inverse of each function.

1. $y = 3x - 2$
2. $y = \frac{x+1}{4}$
3. $y = \frac{1}{3}x + 2$
4. $y = x^3 + 1$
5. $y = 1 + \sqrt{x+5}$
6. $y = 3(x+2)^2 - 7$
7. $y = 2\sqrt{x-1} + 3$
8. $y = \frac{1}{2+x}$
9. $y = \log_3(x+2)$

Answers:

1. $y = \frac{x+2}{3}$
2. $y = 4x - 1$
3. $y = 3x - 6$
4. $y = \sqrt[3]{x-1}$
5. $y = (x-1)^2 - 5$
6. $y = -2 \pm \sqrt{\frac{x+7}{3}}$
7. $y = \left(\frac{x-3}{2}\right)^2 + 1$
8. $y = \frac{1}{x} - 2$
9. $y = 3^x - 2$

Checkpoint 10

Problem 10-176

Rewriting Expressions with and Solving Equations with Logarithms

Answers to problem 10-176: a: $\log_2(5x)$, b: $\log_2(5x^2)$, c: 17, d: $-\frac{9}{20} = 0.45$, e: 15, f: 4

First a review of the properties based on the inverse relationship of exponentials and logarithms:

The following definitions and properties hold true for all positive $m \neq 1$.

Definition of logarithms: $\log_m(a) = n$ means $m^n = a$

Product Property: $\log_m(a \cdot b) = \log_m(a) + \log_m(b)$

Quotient Property: $\log_m(\frac{a}{b}) = \log_m(a) - \log_m(b)$

Power Property: $\log_m(a^n) = n \cdot \log_m(a)$

Inverse relationship: $\log_m(m)^n = n$ and $m^{\log_m(n)} = n$

Example 1: Write as a single logarithm: $3\log_5(x) + \log_5(x+1)$

Solution: $3\log_5(x) + \log_5(x+1) = \log_5(x^3) + \log_5(x+1)$ by the Power Property

$= \log_5(x^3(x+1))$ by the Product Property

$= \log_5(x^4 + x^3)$ by simplifying

Example 2: Solve $\log_2(x) - \log_2(3) = 4$

Solution: $\log_2(x) - \log_2(3) = 4$ problem

$\log_2\left(\frac{x}{3}\right) = 4$ by the Quotient Property

$2^4 = \frac{x}{3}$ by the definition of logarithms

$48 = x$ multiply both sides by 3

Now we can go back and solve the original problems.

a. $\log_2(30x) - \log_2(6)$

$\log_2(\frac{30x}{6}) = \log_2(5x)$

b. $2\log_3(x) + \log_3(5)$

$\log_3(x^2) + \log_3(5)$

$\log_3(x^2 \cdot 5) = \log_3(5x^2)$

c. $\log_7(3x-2) = 2$

$7^2 = 3x - 2$

$49 = 3x - 2 \Rightarrow x = 17$

d. $\log(2x+1) = -1$

$10^{-1} = 2x + 1$

$\frac{1}{10} = 2x + 1 \Rightarrow x = -\frac{9}{20} = 0.45$

e. $\log_5(3y) + \log_5(9) = \log_5(405)$

$\log_5(3y \cdot 9) = \log_5(405)$

$\log_5(27y) = \log_5(405)$

$27y = 405 \Rightarrow y = 15$

f. $\log(x) + \log(x+21) = 2$

$\log(x^2 + 21x) = 2$

$x^2 + 21x = 10^2$

$x^2 + 21x - 100 = 0$

$(x+25)(x-4) = 0$

$x = -25,\ x = 4$

-25 is an extraneous solution so $x = 4$ only

Here are some more to try. In problems 1 through 8, write each expression as a single logarithm. In problems 9 through 26 solve each equation.

1. $\log_3(5)+\log_3(m)$	2. $\log_2(q)-\log_2(z)$
3. $\log_6(r)+3\log_6(x)$	4. $\log(90)+\log(4)-\log(36)$
5. $\log_4(16x)-\log_4(x)$	6. $\log(\sqrt{x})+\log(x^2)$
7. $\log_5(\sqrt{x})+\log_5(\sqrt[3]{x})$	8. $\log_5(x-1)+\log_5(x+1)$
9. $\log_4(2x+3)=\frac{1}{2}$	10. $\log_5(3x+1)=2$
11. $\log_9(9^2)=x$	12. $16^{\log_{16}(5)}=y$
13. $8^{\log_8(x)}=3$	14. $\log_5(5^{0.3})=y$
15. $\log_2(x)=3\log_2(4)+\log_2(5)$	16. $\log_6(x)+\log_6(8)=\log_6(48)$
17. $\log_2(144)-\log_2(x)=\log_2(9)$	18. $\log_2(36)-\log_2(y)=\log_2(12)$
19. $\log_5(3x-1)=-1$	20. $\log_2(x)-\log_2(3)=4$
21. $\frac{1}{3}\log(3x+1)=2$	22. $\log_2(5)+\frac{1}{2}\log_2(x)=\log_2(15)$
23. $\frac{1}{2}\log(y)=2\log(2)+\log(16)$	24. $\log_2(x^2+2x)=3$
25. $2\log_4(x)-\log_4(3)=2$	26. $\log_7(x+1)+\log_7(x-5)=1$

Answers:

1. $\log_3(5m)$	2. $\log_2(\frac{q}{z})$	3. $\log_6(rx^3)$	4. $\log(10)=1$
5. $\log_4(16)=2$	6. $\log(x^{5/2})$	7. $\log_5(x^{5/6})$	8. $\log_5(x^2-1)$
9. $-\frac{1}{2}$	10. 8	11. 2	12. 5
13. 3	14. 0.3	15. 320	16. 6
17. 16	18. 3	19. $\frac{2}{5}$	20. 48
21. 333,333	22. 9	23. 4096	24. −4, 2
25. $\sqrt{48}=4\sqrt{3}$	26. 6		

Checkpoint 11

Problem 11-95

Solving Rational Equations

Answers to problem 11-95: a: $x = \pm 2\sqrt{3}$, b: $x = 2$, c: $x = \frac{2}{9}$, d: $x = \frac{-1 \pm \sqrt{13}}{6} \approx 0.434$ or -0.768

To solve rational equations (equations with fractions) it is usually best to first multiply everything by the common denominator to remove the fractions, a method known as **Fraction Busters**. After you have done this, you can solve the equation using your usual strategies. Following are a few examples.

Example 1: Solve $\frac{24}{x+1} = \frac{16}{1}$.

Solution: The common denominator in this case is $(x+1)$. Multiplying both sides of the equation by $(x+1)$ removes all fractions from the equation. You can then simplify and solve for x. This process is demonstrated at right.

$$(x+1)\left(\frac{24}{x+1}\right) = (x+1)\left(\frac{16}{1}\right)$$
$$24 = 16(x+1)$$
$$24 = 16x + 16$$
$$8 = 16x$$
$$x = \frac{1}{2}$$

Answer: $x = \frac{1}{2}$

Example 2: Solve $\frac{5}{2x} + \frac{1}{6} = 8$.

Solution: Again, we multiply both sides of the equation by the common denominator, which, in this case, is $6x$. We must be careful to remember to distribute so that we multiply each term on both sides of the equation by $6x$. Then we simplify and solve, as shown at right.

$$6x\left(\frac{5}{2x} + \frac{1}{6}\right) = 6x(8)$$
$$6x \cdot \frac{5}{2x} + 6x \cdot \frac{1}{6} = 48x$$
$$15 + x = 48x$$
$$15 = 47x$$
$$x = \frac{15}{47}$$

Answer: $x = \frac{15}{47}$

Now we can go back and solve the original problems.

a.

$\frac{x}{3}=\frac{4}{x}$

$3x\left(\frac{x}{3}\right)=3x\left(\frac{4}{x}\right)$

$x^2=12$

$x=\pm\sqrt{12}=\pm2\sqrt{3}$

$x\approx\pm3.46$

b.

$\frac{x}{x-1}=\frac{4}{x}$

$x(x-1)\left(\frac{x}{x-1}\right)=x(x-1)\left(\frac{4}{x}\right)$

$x^2=4(x-1)$

$x^2-4x+4=0$

$(x-2)(x-2)=0$

$x=2$

c.

$\frac{1}{x}+\frac{1}{3x}=6$

$3x\left(\frac{1}{x}+\frac{1}{3x}\right)=3x(6)$

$3x\left(\frac{1}{x}\right)+3x\left(\frac{1}{3x}\right)=18x$

$3+1=18x$

$x=\frac{2}{9}$

d.

$\frac{1}{x}+\frac{1}{x+1}=3$

$x(x+1)\left(\frac{1}{x}+\frac{1}{x+1}\right)=x(x+1)(3)$

$x(x+1)\left(\frac{1}{x}\right)+x(x+1)\left(\frac{1}{x+1}\right)=x(x+1)(3)$

$x+1+x=3x^2+3x$

$0=3x^2+x-1$

Using the Quadratic Formula,
$x\approx-0.434,-0.768$

Here are some more to try. Solve each of the following rational equations.

1. $\frac{3x}{5}=\frac{x-2}{4}$
2. $\frac{4x-1}{x}=3x$
3. $\frac{2x}{5}-\frac{1}{3}=\frac{137}{3}$
4. $\frac{4x-1}{x+1}=x-1$
5. $\frac{x}{3}=x+4$
6. $\frac{x-1}{5}=\frac{3}{x+1}$
7. $\frac{x+6}{3}=x$
8. $\frac{2x+3}{6}+\frac{1}{2}=\frac{x}{2}$
9. $\frac{3}{x}+\frac{5}{x-7}=-2$
10. $\frac{2x+3}{4}-\frac{x-7}{6}=\frac{2x-3}{12}$

Answers:

1. $x=-\frac{10}{7}$
2. $x=\frac{1}{3},1$
3. $x=115$
4. $x=0,4$
5. $x=-6$
6. $x=\pm4$
7. $x=3$
8. $x=6$
9. $x=\frac{3\pm\sqrt{51}}{2}$
10. $x=-13$